JUST-NON-CROSS VARIETIES
OF GROUPS

by

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The results presented in this thesis are mine except where otherwise stated.

J M Brady

J.M. Brady.

PREFACE

I wish to thank:

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CHAPTER 1

INTRODUCTION

1.1 Statement of the problem

In 1937, B.H. Neumann [17] asked whether the laws of (the variety generated by) a finite group have a finite basis. This problem was taken up over twenty years later by one of Graham Higman's research students, D.C. Cross, and in a report [7] on the latter's progress to the International Convention on the Theory of Finite Groups in Florence in 1960, Higman pointed out the relevance of locally finite varieties having a finite basis for their laws, and containing only finitely many (isomorphism classes of) critical groups. Such varieties have since come to be called Cross varieties [18, 51.51]. The efforts in Oxford reached fruition in 1964 with the celebrated Theorem of Shiela Oates and M.B. Powell [20], which may be stated thus: Cross varieties are precisely those that can be generated by a single finite group. (An alternative version of this result, due to L.G. Kovács and M.F. Newman, will be quoted later as Theorem 2.4.2.) A corollary to the Oates-Powell Theorem is the fact that the set of Cross varieties forms a sublattice C of the lattice L of all varieties of groups.

Kovács and Newman [14] have suggested that worthwhile information about L-C, and some insight into the general finite basis problem, might be gained by classifying those elements of L which are minimal with respect to not belonging to C. They have since shown [15, Theorem 1] that all non-Cross varieties (i.e. elements of $L-C$) contain one of these minimal elements (which they call just-non-Cross varieties) as a subvariety. In the remainder of this thesis, "just-non-Cross" will be abbreviated to "jnC", as this is a contribution to the problem of classifying jnC varieties.

We need to describe the main result of Kovács and Newman [14]; but before we can, we shall have to develop some notation. Let p be a prime, and n a positive integer. The variety consisting of all abelian groups (respectively of all groups of exponent dividing n) is $\underline{\underline{A}}$ (respectively $\underline{\underline{B}}_n$), and the meet of $\underline{\underline{A}}$ and $\underline{\underline{B}}_n$ is $\underline{\underline{A}}_n$. We denote the variety consisting of all nilpotent groups of class (at most) two by $\underline{\underline{N}}_2$. If p is odd, Q_p is the nonabelian group of exponent p and order p^3; while Q_2 is the quaternion group of order eight. In either case, we denote the subvariety of $\underline{\underline{N}}_2$ generated by Q_p by $\underline{\underline{T}}_p$. The class consisting of all locally finite groups of exponent p is a variety (L.G. Kovács [11] and A.I. Kostrikin [10]), denoted by $\underline{\underline{K}}_p$.

Kovács and Newman [14, page 221] observe first that $\underline{\underline{A}}$ is jnC. Moreover, as any variety of infinite exponent must contain an infinite cyclic group (which generates $\underline{\underline{A}}$), it is the *only* jnC variety of infinite exponent. Of course it is also the only abelian jnC variety.

Call a variety decomposable in case it has proper subvarieties $\underline{\underline{V}}_1$ and $\underline{\underline{V}}_2$ such that $\underline{\underline{V}}$ is their product $\underline{\underline{V}}_1\underline{\underline{V}}_2$. The main result of [14] is:

1.1.1 THEOREM The decomposable jnC varieties are precisely the $\underline{\underline{A}}_p\underline{\underline{A}}_p$, $\underline{\underline{A}}_p\underline{\underline{A}}_q\underline{\underline{A}}_r$, $\underline{\underline{A}}_p\underline{\underline{T}}_q$, where p, q and r are any three distinct primes. In particular, the decomposable jnC varieties are all soluble of finite exponent.

There is also a conjecture in [14], namely:

1.1.2 CONJECTURE. Every soluble jnC variety of finite exponent is decomposable.

1.2 Statement of results

The main Theorem of this thesis (Theorem C) has the following consequences:

THEOREM A The soluble jnC varieties which are not abelian-by-nilpotent are the $\underline{\underline{A}}_p\underline{\underline{A}}_q\underline{\underline{A}}_r$, where p, q and r are any three distinct primes.

THEOREM B The non-metabelian jnC varieties which are abelian-by-nilpotent of class two are the $\underline{\underline{A}}_p\underline{\underline{T}}_q$, where p and q are distinct primes.

The crucial property of soluble jnC varieties of finite exponent is that they are reducible in the following sense: a variety $\mathbf{V}$ is called <u>reducible</u> if it has proper subvarieties $\mathbf{V}_1$ and $\mathbf{V}_2$ such that $\mathbf{V}$ is a subvariety of $\mathbf{V}_1\mathbf{V}_2$.

THEOREM A* The reducible jnC varieties which are not abelian-by-nilpotent are the $\mathbf{A}_p\mathbf{A}_q\mathbf{A}_r$, where p, q and r are any three distinct primes.

In section 1.3, we shall derive as corollaries to Theorems A and A* some results of John Cossey and L.G. Kovács about locally finite jnC varieties in which the nilpotent groups are all abelian. Our main result may be stated as follows:

THEOREM C If there is a reducible jnC variety $\mathbf{V}$ which is not decomposable, then there are distinct primes p and q and a positive integer n (all three depending on $\mathbf{V}$), such that $\mathbf{V}$ is a subvariety of $\mathbf{A}_p(\mathbf{V} \wedge \mathbf{B}_{q^n})$, and $\mathbf{V} \wedge \mathbf{B}_{q^n}$ must be nilpotent of class at least three.

1.2.1 COROLLARY A jnC variety is reducible if and only if it is soluble of finite exponent.

Theorem C is phrased so as to emphasize the fact that its hypotheses can be satisfied (if and) only if Conjecture 1.1.2 is false. Consider the following special case of Conjecture 1.1.2:

1.2.2 CONJECTURE If $\underline{\underline{N}}$ is a nilpotent variety of class at least two and exponent a power of the prime q, and if p is a prime unequal to q, then $\underline{\underline{A}}_p\underline{\underline{T}}_q$ is the unique jnC subvariety of $\underline{\underline{A}}_p\underline{\underline{N}}$.

1.2.3 COROLLARY Conjecture 1.1.2 is true if and only if Conjecture 1.2.2 is.

To date, the only significant progress we have made with Conjecture 1.2.2 is Theorem B, which shows it to be true in case $\underline{\underline{N}}$ has class two. Even in case $\underline{\underline{N}}$ has class three and exponent q, we have not got very far; our progess is reported in section 6.2.

Apart from the gap left by our inability to decide Conjecture 1.2.2, Theorem C reduces the classification problem to finding the irreducible jnC varieties. This is discussed more fully in section 6.1, where for example it is shown (Corollary 6.1.3) that a jnC variety $\underline{\underline{V}}$ is irreducible if and only if either a) $\underline{\underline{V}}$ is not locally finite, or, b) $\underline{\underline{V}}$ is locally finite and locally nilpotent but insoluble, or c) $\underline{\underline{V}}$ is locally finite and contains infinitely many (isomorphism classes of) finite simple groups.

It may be in order to comment briefly on each of these three possible kinds of irreducible jnC varieties. Of course, $\underline{\underline{A}}$ is irreducible of type (a); what is not known, however, is whether there are any irreducibles of finite exponent of type (a). The recently announced result [1] of Seymour Bachmuth, Horace Y. Mochizuki and David Walkup implies that $\underline{\underline{K}}_5$ is non-Cross, and hence that there is an irreducible of type (b). As regards those of type (c), Graham Higman points out [8, page 154] "that a locally finite variety can contain only a finite number of [isomorphism classes of] finite simple groups ... would follow in turn either from the conjecture that there are only a finite number of finite simple groups of given exponent, or from the conjecture that there is a bound to the number of elements necessary to generate a finite simple group".

I should like to conclude this section by stating that, in my opinion, the difficulties liable to be encountered in filling any of the gaps remaining in the classification are considerable.

1.3 A history of the project

We begin this section by surveying those classification results, apart from Theorem 1.1.1, which were known at the start of this project, and those that became available during it. A (then unpublished) theorem of L.G. Kovacs and M.F. Newman [15, Theorem 5] implies that the soluble locally nilpotent jnC varieties of finite exponent are the $\underline{\underline{A}}_p\underline{\underline{A}}_p$, for all primes p. Again, the main result of John Cossey's thesis [3] says that the soluble locally finite jnC varieties in which the nilpotent groups are all abelian ("the jnC varieties of s$\underline{\underline{A}}$-groups") are precisely the $\underline{\underline{A}}_p\underline{\underline{A}}_q\underline{\underline{A}}_r$, where p,q and r are any three distinct primes. Cossey and Kovács conjectured that in this the assumption of solubility was unnecessary; during 1968, Kovács proved this using a lemma of A.Yu.Ol'šanskij [21, Lemma 5]. He was then able to apply this extended version of Cossey's result to give a routine proof of the main result of [21]; namely, that every group in a variety $\underline{\underline{V}}$ is residually finite if and only if $\underline{\underline{V}}$ can be generated by a finite group whose Sylow subgroups are all abelian.

One suspected originally that Kovács's argument to generalise Cossey's result was itself capable of substantial generalisation. Strictly speaking, this has not been the case. Instead, by replacing most of that proof by other considerations (which restricted to the case treated by him would be substantially simpler than the originals), I have been able to prove Theorem C.

To conclude this chapter, we briefly sketch derivations of the results of Cossey and Kovács from Theorems A and A* respectively.

Cossey has shown [3, 4.1.1] that a metabelian variety of s$\underline{\underline{A}}$-groups is Cross. Let $\underline{\underline{V}}$ be a jnC variety of s$\underline{\underline{A}}$-groups. If $\underline{\underline{V}}$ were abelian-by-nilpotent, it would be metabelian, and hence Cross, a contradiction. It follows from Theorem A that $\underline{\underline{V}}$ is an $\underline{\underline{A}}_p\underline{\underline{A}}_q\underline{\underline{A}}_r$.

Let $\underline{\underline{V}}$ be a locally finite jnC variety in which the nilpotent groups are all abelian. If $\underline{\underline{V}}$ were abelian-by-nilpotent, it would be a metabelian variety of s$\underline{\underline{A}}$-groups, and we would have a contradiction as before. It suffices, therefore, by Theorem A* to show that $\underline{\underline{V}}$ is reducible. But as we shall show as Theorem 6.1.2, an irreducible locally finite variety is either generated by a finite simple group, or is locally nilpotent but insoluble, or it contains infinitely many pairwise-nonisomorphic finite simple groups. In the present case, the first two options cannot hold, and the third is ruled out by J.H. Walter's results on finite simple groups with abelian Sylow 2-subgroups [22].

It may be helpful to the reader if we provide him with a travel guide, so that he may more easily find his way around this thesis. The four main results proved in the text are Theorem 3.2.1, Corollary 4.3.3, and Theorems 5.1.1 and 5.1.2 (in Chapters 3, 4 and 5 respectively). The deductions of Theorems A , A* , B and C from these results are given in section 5.1.

CHAPTER 2

PRELIMINARIES

As is so often the case in writing Mathematics, one of the more difficult problems I have faced in the production of this thesis has been that of devising a notation which would be simple enough not to lead to confusion, and yet which would be rich enough to contend with the various structures encountered; namely groups, permutation groups, representations of groups, vector spaces, varieties, classes, fields, sets of numbers, and sets in general. In section 2.1, I describe in general terms my solution to the problem. In each of the other sections, I fix on one (or more) of the items in the above list, expand the notation suitably, and then give some technical results for use in the main body of the thesis (Chapters 3, 4 and 5). Those results for which no proof is offered are usually well-known and are included for easier reference.

2.1 Notation

We use capital Roman letters for groups and vector spaces, and small Roman letters for their elements. Every trivial group is denoted by E, and the identity of every group by e. Every trivial vector space is denoted by 0, and the neutral element of every vector space by o. We follow the established practice, and denote varieties of groups, and classes generally, by capital Roman letters with a double underline. (In this way we simulate capital German letters.) Our Roman hierarchy is complete when we denote fields and sets of numbers (by which we mean subsets of the field of real numbers) by capital Roman letters with a single underline. (In this way we simulate bold-face type.) In particular, the Galois field of p elements is $\underline{F}(p)$, the rationals $\underline{Q}$, and the reals $\underline{R}$. The set of natural numbers (in which we include zero) is denoted by $\underline{N}$, and the set of natural numbers greater than zero is $\underline{P}$. The multiplicative identity element of a field will always be 1. If $m,n \in \underline{N}$, $m \geq n$, we denote the binomial coefficient $m!/n!m-n!$ by $\binom{m}{n}$. If $x \in \underline{R}$, $x \geq 0$, $[x]$ is the unique element of $\underline{N}$ such that $[x] \leq x < [x] + 1$. *The letters* p,q *and* r *will always denote primes.* Apart from those sets whose notation has already been specified, most sets are denoted by capital Greek letters; an exception being the empty set $\emptyset$. The possible ambiguity between sets and classes inherent in this framework will always be unimportant. As well as denoting elements of sets, small Greek letters will be used for functions; this includes group representations.

The end of a proof will be signified by writing "//". In case this immediately follows the statement of a result, we imply that no proof is offered.

2.2 Groups and permutation groups

Throughout this thesis, "group" means "finite group", except in certain places, when the meaning will always be clear from the context. Sometimes (e.g. in the statement "a locally finite variety is generated by its finite groups"), the word "finite" will be included for emphasis.

If G is a group, and H is a subgroup of G, we write $H \leq G$; if $H \leq G$, but $H \neq G$, we write $H < G$. The index of H in G is $|G:H|$, and we denote the centraliser (normaliser) of H in G by $C_G(H)(N_G(H))$. If Λ is a set whose elements are either elements of G, subsets of G, or subgroups of G, (or a mixture of all three), we denote the subgroup of G generated by Λ by $<\Lambda>$. For elements g,h of G, the conjugate $h^{-1}gh$ of g by h is denoted by g^h, and $g^{-1}g^h$ by $[g,h]$. From the latter we get inductively left-normed commutators of weight $c+1$, $c>2$, as follows:

$$[x_1,\ldots,x_c,x_{c+1}] = [[x_1,\ldots,x_c],x_{c+1}];$$

we shall denote $[g,h,h]$ by $[g,2h]$.

A subgroup H of G is a central factor if $G = <H,C_G(H)>$. The group G is a central product of its subgroups $H_1,\ldots,H_k$ if $G = <H_1,\ldots,H_k>$, and if, for all i, $C_G(H_i) \geq <H_j:1\leq j\leq k, j\neq i>$; in particular, each H_i is a central factor of G. Central products will be discussed extensively in section 4.1. If K

is a normal subgroup of H, $H \leq G$, we call the quotient group H/K a section of G. The section H/K of G is a chief section in case K is normal in G and H/K is a minimal normal subgroup of G/K. We remark that the word "section" is preferred to the more commonly used "factor" to avoid confusion between central section and central factor.

The soluble radical (soluble residual) of G is the normal subgroup of G maximal (minimal) with respect to being soluble (to having soluble quotient group). The socle $M(G)$ of a group G is the subgroup of G generated by all the minimal normal subgroups of G. Since we are dealing only with finite groups, $M(G) = E$ if and only if $G = E$. A group G is monolithic in case it has precisely one minimal normal subgroup. The k^{th} term of the derived series (lower central series) of a group G is $G^{(k)}$ ($\underline{\underline{N}}_k(G)$), so that $G = G^{(0)} = \underline{\underline{N}}_0(G)$. We say that G is perfect in case $G = G^{(1)}$. The centre of G is $Z(G)$, and the hypercentre $Z_\infty(G)$; the Frattini subgroup is $D(G)$, and $d(G)$ is the minimum of the cardinalities of generating sets of G. As is well-known, if G is a p-group, $p^{d(G)} = |G : D(G)|$. Finally, the automorphism group of G is $\mathrm{Aut}G$.

The following trivial facts will be used often:

2.2.1 LEMMA (i) If N is a normal subgroup of G, and $N \cap G^{(1)} = E$, then $N \leq Z(G)$.

(ii) If $N_1, \ldots, N_k$ are normal subgroups of G, the quotient group $G/\cap\{N_i : 1 \leq i \leq k\}$ can be embedded as a subgroup in the direct product $G/N_1 \times \ldots \times G/N_k$.

(iii) If $G/Z(G)$ is perfect, so is $G^{(1)}$. //

We next recall that a subgroup T of a group B is <u>intravariant</u> in B if the image of T under every automorphism of B is conjugate to T in B. Thus a normal subgroup is intravariant if and only if it is characteristic. In this direction, we need a couple of lemmas (for use in section 3.1), which we preface with the following remark:

2.2.2 LEMMA Let T be an intravariant subgroup of a group B. Then $C_B(T)$ and $N_B(T)$ are also intravariant subgroups of B. If U is an intravariant subgroup of T, U is intravariant in B. //

2.2.3 LEMMA A nonabelian simple group has a non-nilpotent intravariant proper subgroup.

Proof. Let B be a nonabelian simple group, and let r be any odd prime dividing $|B|$. Sylow's Theorems assert that a Sylow r-subgroup R of B is intravariant in B. By Lemma 2.2.2, $C_B(Z(R))$ and $N_B(J(R))$ are intravariant (necessarily proper) subgroups of B, where $J(R)$ denotes the Thompson subgroup of R (see [9, iv, 6.1]). By a theorem of J.G. Thompson [9, iv, 6.2], at least one of $C_B(Z(R))$ and $N_B(J(R))$ is not even r-nilpotent. //

2.2.4 LEMMA Let B be a p'-subgroup of a group G, and let T be an intravariant subgroup of B. If P is a p-subgroup of $N_G(B)$, then P also normalises some B-conjugate of T.

Proof. Let the distinct B-conjugates of T be $T = T_1, T_2, \ldots, T_n$. Since $n = |B : N_B(T)|$, p does not divide n. Since P normalises B, and T is intravariant in B, P permutes (by conjugation) $\{T_1, \ldots, T_n\}$. But (see Lemma 2.2.7(i)) an orbit of P has cardinality a power of p, so that P has a fixed point, say T_i, in $\{T_1, \ldots, T_n\}$. That is, $P \leq N_G(T_i)$. //

We conclude this section by considering the following situation. Let G be a group, and let N be a minimal normal subgroup of G. Conjugation by elements of G induces automorphisms of N; in this way G is represented as a subgroup of AutN with kernel $C_G(N)$.

The proofs of the major theorems of this thesis derive essentially from closer analyses of such representations. We now describe how we are led to study group representations (in the sense of C. Curtis and I. Reiner [4]) when N is abelian, and permutation representations when N is nonabelian.

If N is abelian, say of exponent p, we can think of it as a vector space over $\underline{F}(p)$; thus conjugation by elements of G induces linear transformations on N, and since $N \leq C_G(N)$, $G/C_G(N)$ is faithfully and irreducibly represented on the $\underline{F}(p)$-space N. We shall return to group representations in the next section.

In case N is nonabelian, there is a nonabelian simple group, say B, such that N is isomorphic to a direct power of B. We need the following lemma:

2.2.5 LEMMA Let B be a group with trivial centre, and let K be a normal subgroup of the direct product $B_1 \times \ldots \times B_n$, where each B_i is isomorphic to B. Then $K \cap B_i$ is nontrivial if and only if the image of K under its projection into B_i is nontrivial. In particular, if B is a nonabelian simple group, K is the direct product of some subset of $\{B_1,\ldots,B_n\}$.

Proof. Denote the projection of K into B_i by π_i. If $K\pi_i$ is nontrivial, there is an element, say k, in K with $k\pi_i \neq e$.

Since $Z(B_i) = E$, there is an element, say b_i, of B_i which fails to commute with k^{π_i}. Then $[k,b_i]$ is a nonidentity element of $K \cap B_i$. //

2.2.6 COROLLARY Let the simple direct factors of the nonabelian minimal normal subgroup N of G be $B_1,\ldots,B_n$. Then if $g \in G$, and $1 \leq i \leq n$, $B_i^g \in \{B_1,\ldots,B_n\}$. Also, G is represented by conjugation as a transitive permutation group on $\{B_1,\ldots,B_n\}$.

Proof. Since B_i^g is a normal subgroup of N, and is isomorphic to B_i, the first statement follows immediately from Lemma 2.2.5. Since N is a minimal normal subgroup of G, the normal closure of B_1 in G is N. Hence the orbit of G containing B_1 is $\{B_1,\ldots,B_n\}$, and G is transitive. //

Thus we are led to consider permutation groups. We shall consider only permutation groups acting on finite sets, and so, for convenience, *we take for granted all of Chapter 1 of Helmut Wielandt's book* [23]. If G is a permutation group acting on a set Ω, the image of a point $\alpha \in \Omega$ under an element $g \in G$ is α^g. The stabiliser of α will be denoted by G_α, and the orbit of α under a subset S of G by α^S. We shall need the following facts several times:

2.2.7 LEMMA Let G be a permutation group on a set Ω.

(i) For all $\alpha \in \Omega$, $|G| = |G_\alpha|.|\alpha^G|$. In particular, if G is a p-group, $|\alpha^G|$ is a power of p. [23, 3.2].

(ii) If P is a Sylow p-subgroup of G, then for all points α in Ω, $|\alpha^P|$ is at least the p-share of $|\alpha^G|$. [23, 3.4]. //

2.2.8 LEMMA Let G be a permutation group on a set Ω, H a subgroup of G, and g an element of $N_G(H)$. If $\alpha \in \Omega$, $\alpha^H = \alpha^{\langle H,g\rangle}$ if and only if $\alpha^H \cap \alpha^{Hg}$ is non-empty.

Proof. The "only if" part is easy, so suppose that $\alpha^H \cap \alpha^{Hg}$ is non-empty. Then there is a point $\beta \in \alpha^H$ such that $\beta^g \in \alpha^H$. Let γ be *any* point of α^H; then for some element $h \in H$, $\beta^h = \gamma$. Thus

$$\gamma^g = \beta^{hg} = \beta^{gh[h,g]} \in \alpha^H,$$

since $[h,g] \in H$. It follows that $\alpha^{Hg} = \alpha^H$, and so for all $n \in \underline{N}$, $\alpha^{Hg^n} = \alpha^H$. But if $g^m \in H$, every element of $\langle H,g\rangle$ can be written in the form hg^n, for some $n \leq m$. Thus $\alpha^{\langle H,g\rangle} = \alpha^H$. //

2.2.9 THEOREM Let G be a group, and suppose that the subgroup M of G is the direct product of its subgroups $B_1,\ldots,B_{p^t}$. Let P be a p-subgroup of G, and suppose that conjugation by elements of P transitively permutes $B_1,\ldots,B_{p^t}$. If $P \cap B_1$ is nontrivial, P has class at least $t + 1$.

Proof. As the claim is trivial for $t = 0$, we suppose $t > 0$. Put Ω equal to $\{B_1,\ldots,B_{p^t}\}$, and denote $\bigcap\{N_p(B_i) : 1 \leq i \leq p^t\}$ by N and $N_p(B_1)$ by P_0. Then P/N acts (by conjugation) as a transitive permutation group on Ω, the stabiliser of the "point" B_1 being P_0/N. It follows from Lemma 2.2.7 (i) that $|P : P_0| = p^t$. Now a maximal subgroup of a p-group is normal, and has index p, and so P has subgroups $P_1,\ldots,P_t$ such that

$$P_0 \lhd P_1 \lhd \ldots \lhd P_t = P.$$

Let x_i be an element of $P_i - P_{i-1}$, $1 \leq i \leq t$. Since $|P_i : P_0| = p^i$, Lemma 2.2.7 (i) implies that an orbit of P_i has cardinality p^i. If S is a subset of P, $\langle B_1^{SN/N}\rangle$ is the direct product of the elements of $\{B_1^g : g \in S\}$, and so we may assume that the points of Ω have been numbered so that

$$\langle B_1^{P_i/N}\rangle = B_1 \times \ldots \times B_{p^i},$$

$0 \le i \le t$. Since $P_{i+1} = \langle P_i, x_{i+1}\rangle$, it follows from Lemma 2.2.8 that $B_1^{P_i/N} \cap B_1^{P_i x_{i+1} N/N}$ is empty. Let x_0 be an element of B_1, $e \neq x_0$. We shall prove by induction on i that (for $0 \le i \le t$) $e \neq [x_0,\ldots,x_i] \in \langle B_1^{P_i/N}\rangle$. In case $i = 0$, this claim reduces to $e \neq x_0 \in B_1$. Suppose that $i < t$, and that $e \neq [x_0,\ldots,x_i] \in \langle B_1^{P_i/N}\rangle$. Since $\langle P_i, x_{i+1}\rangle = P_{i+1}$, it follows that $[x_0,\ldots,x_{i+1}] \in \langle B_1^{P_{i+1}/N}\rangle$. Now

$$[x_0,\ldots,x_{i+1}] = [x_0,\ldots,x_i]^{-1}[x_0,\ldots,x_i]^{x_{i+1}} ,$$

and $e \neq [x_0,\ldots,x_i] \in \langle B_1^{P_i/N}\rangle$; thus $e \neq [x_0,\ldots,x_i]^{x_{i+1}} \in \langle B_1^{P_i x_{i+1} N/N}\rangle$. Since $B_1^{P_i/N} \cap B_1^{P_i x_{i+1} N/N}$ is empty, the claim is established. In particular, if $P \cap B_1$ is nontrivial, we can choose $x_0 \in P$. Since $[x_0,\ldots,x_t] \neq e$, it then follows that P has class at least $t + 1$. //

We shall need two corollaries to Theorem 2.2.9; however the second of these must be deferred until we have discussed Clifford's Theorem in section 2.3.

2.2.10 COROLLARY Let the minimal normal subgroup M of G be isomorphic to a direct product of p^t copies of the nonabelian simple group B. If P is a Sylow p-subgroup of G, and $M \cap P$ is nontrivial, then P has class at least $t + 1$.

Proof. By Corollary 2.2.6 and Lemma 2.2.7 (ii), conjugation by elements of P transitively permutes the simple direct factors of M. If $M \cap P$ is nontrivial, P intersects each simple direct factor of M nontrivially, and the result follows from Theorem 2.2.9. //

2.3 Representations of groups

If $\underline{E}$ is a field, and $n \in \underline{P}$, we denote by $GL(n,\underline{E})$ the group of all $n \times n$ nonsingular matrices over $\underline{E}$. Similarly, if V is an n-dimensional vector space over $\underline{E}$, the group of all bijective linear transformations from V to V is $GL(V)$. We recall that if G is a group, a representation τ of G of dimension (or degree) n over $\underline{E}$ is a (group) homomorphism $\tau : G \to GL(n,\underline{E})$. We shall assume that the connection between representations of G over $\underline{E}$ and finite-dimensional right $\underline{E}G$-modules is known. As we only consider finite-dimensional right modules, we always omit the qualifying adjectives. We shall further assume familiarity with the concept of the tensor product $V \otimes_{\underline{E}} W$ of two $\underline{E}$-spaces V and W, and some of its elementary properties. To be more explicit: *we take for granted all of Chapter 2 of the book* [4] *by C.W. Curtis and I. Reiner.* We shall use " $\oplus$ " to indicate that a sum of vector spaces is a direct sum; otherwise we shall use "$\sum$" for sums. In the statement "$\underline{E}$ is a field whose characteristic does not divide $|G|$", we allow for the possibility that $\underline{E}$ has characteristic zero. If $H \leq G$, and V is an $\underline{E}G$-module, the $\underline{E}H$-module obtained from V by restriction is denoted by V_H. Similarly, if τ is a representation of G, the representation of H obtained by restricting τ to H is denoted by τ_H.

The following theorem [4,10.8] is used extensively, and will be referrred to as "Maschke's Theorem".

MASCHKE'S THEOREM If G is a group, and $\underline{E}$ is a field whose characteristic does not divide $|G|$, every representation of G over $\underline{E}$ is completely reducible.

2.3.1 THEOREM Let B be a nonabelian group, $\underline{E}$ a field, and V a faithful $\underline{E}B$-module. In case $\underline{E}$ has (nonzero) characteristic p, suppose that $B^{(1)}$ is not a p-group. Then there is an abelian-by-cyclic subgroup S of B such that V_S has an irreducible submodule of dimension at least two.

Proof. Suppose $\underline{E}$ has (possibly zero) characteristic p, and choose any prime q unequal to p which divides $|B|$. (Thus if p is zero, q may be chosen to be any prime dividing $|B|$.) Let Q be any abelian q-subgroup of B, and let $g \in N_B(Q)$. Denote $\langle Q,g\rangle^{(1)}$ by R; then $R \leq Q$, and so by Maschke's Theorem V_R is completely reducible; say,

$$V_R = V_1 \oplus \cdots \oplus V_k$$

is a decomposition of V_R into irreducible submodules. Let W_1 be the sum of those V_i that are trivial, and W_2 the sum of those that are not. Observe that both W_1 and W_2 admit $\langle Q,g\rangle$, and hence that

$$V_{\langle Q,g\rangle} = W_1 \oplus W_2.$$

If W_2 is nonzero for some choice of Q and g, let U be any irreducible $\underline{E}\langle Q,g\rangle$-submodule of it. Now no nonzero element of U is fixed by the whole of R, and so R is nontrivial and the kernel N of U avoids R. It follows that $\langle Q,g\rangle/N$ is nonabelian and is faithfully and irreducibly represented on U; thus U has $\underline{E}$-dimension at least two. In this case, therefore, we can choose S equal to $\langle Q,g\rangle$.

Suppose, on the other hand, that for all possible choices of Q and g, W_2 is zero. Since V is faithful, it follows that R

is always trivial, and hence that $N_B(Q) = C_B(Q)$ for all abelian q-subgroups Q of G. But in any q-group the subgroups maximal with respect to being abelian and normal are self-centralising [9, III, 7.3]. Since we could have chosen g to be a q-element, we must conclude that every q-subgroup of B is abelian. By a theorem of Burnside [9, IV, 2.6], B is q-nilpotent for all q unequal to p. If p is zero, it follows that B is nilpotent, and hence abelian, a contradiction. If p is not zero, then since the normal q-complement $B(q)$ of B is complemented in B by an (abelian) Sylow q-subgroup of B, $B(q) \geq B^{(1)}$. It follows that $B^{(1)} \leq \cap\{B(q) : q \neq p\}$, and hence that $B^{(1)}$ is a p-group. Again we have a contradiction, and the Theorem is proved. //

In the later chapters of this thesis, we frequently need to quote a rather detailed version of Clifford's Theorem, so it seems worthwhile recalling the basic facts as they are presented, for instance, in B. Huppert's book [9, V, Section 17]: let G be a group, $N \trianglelefteq G$, and V an $\underline{E}G$-module, where $\underline{E}$ is some field. Let U be a submodule of V_N. If $g \in G$, $Ug(= \{ug : u \in U\})$ is a submodule of V_{N^g}, for

$$(ug)g^{-1}ng = (ugg^{-1})ng = (un)g \in Ug.$$

Observe that Ug is irreducible if and only if U is, and that if N is a normal subgroup of G, Ug is in fact a submodule of V_N. Notice also that if K is the kernel of U, that of Ug is K^g.

A completely reducible module is called homogeneous in case it is a sum of isomorphic irreducible modules; the homogeneous components of a completely reducible module are the maximal homogeneous submodules.

CLIFFORD'S THEOREM Let G be a group, N a normal subgroup of G, and V an irreducible $\underline{E}G$-module, where $\underline{E}$ is any field.

(i) If U is an irreducible submodule of V_N, then $V_N = \sum\{Ug : g \in G\}$. In particular, V_N is completely reducible.

(ii) Select elements $g_1,\ldots,g_k$ of G such that $\{Ug_1,\ldots,Ug_k\}$ is a complete set of representatives of the isomorphism types of the irreducible submodules of V_N, and put $V_i = \sum\{Ug : g \in G, Ug \cong Ug_i\}$. Then $V_N = V_1 \oplus \ldots \oplus V_k$, and the V_i are the homogeneous components of V_N. Suppose that V_i is a direct sum of ℓ_i irreducible submodules.

(iii) The inertia group N_i of V_i, $1 \leq i \leq k$, is $\{g \in G : Ug \cong Ug_i\}$; then $N \leq N_i$, V_i is an irreducible submodule of V_{N_i}, and $V_i^G \cong V$.

(iv) For all $g \in G$, for all i, there is a j such that $N_i^g = N_j$ and $V_ig = V_j$; in this way, G acts as a transitive permutation group on $\{V_1,\ldots,V_k\}$. The stabiliser of V_i is N_i, and so $|G : N_i| = k$, $1 \leq i \leq k$. Notice also that since $V_1 g_i = V_i$, $\ell_1 = \ell_2 = \ldots = \ell_k$. //

In connection with Clifford's Theorem, we find it useful to have available the following abbreviation: if G is a group, N is a normal subgroup of G, and V is an irreducible $\underline{E}G$-module, for some field $\underline{E}$, we shall say"

$$V_N = \bigoplus_{i=1}^{k} V_i \; ; \quad V_i = \bigoplus_{j=1}^{\ell} V_{ij}$$

is a Clifford decomposition of V_N" to indicate that $V_1,\ldots,V_k$ are the homogeneous components of V_N, and that

$$V_i = \bigoplus_{j=1}^{\ell} V_{ij}$$

is a direct decomposition of V_i into (isomorphic) irreducible submodules.

We now use Clifford's Theorem to deduce three lemmas. The first will be needed in Chapters 3 and 5, and the second in Chapters 3, 4 and 5. The third lemma is the second corollary to Theorem 2.2.9 promised earlier.

2.3.2 LEMMA Let G be a group, $\underline{E}$ a field, and V an irreducible $\underline{E}G$-module. Let N be a normal subgroup of G, and suppose that V_N has k homogeneous components. If R is a Sylow r-subgroup of G, and U is an irreducible submodule of V_{NR}, then the number of homogeneous components of U_N is at least the r-share of k. In particular, the $\underline{E}$-dimension of U is at least the r-share of k.

Proof. Suppose that $V_1,\ldots,V_k$ are the homogeneous components of V_N. Let H_i be the inertia group of V_i, $1 \leq i \leq k$, and denote $\bigcap_{i=1}^{k} H_i$ by H. By Clifford's Theorem, G/H acts as a transitive permutation group on $\{V_1,\ldots,V_k\}$; the stabiliser of the "point" V_i being H_i/H. Since N is a normal subgroup of NR, it follows from Clifford's Theorem that U_N is completely reducible, and that

$$U_N = (U_N \cap V_1) \oplus \ldots \oplus (U_N \cap V_k)$$

is the decomposition of U_N into its homogeneous components (although we allow for the possibility that some of the $(U_N \cap V_i)$ will be zero). We may suppose without loss of generality that the V_i have been numbered so that $U_N \cap V_1$ is not zero. Since U is irreducible, it is spanned by $(U_N \cap V_1)R$. Hence the number of homogeneous components of U_N is the cardinality of the orbit of RH/H containing V_1, and this is at least the r-share of k (Lemma 2.2.7 (ii). //

2.3.3 LEMMA Let G be a group, $\underline{E}$ a field, and V an irreducible $\underline{E}G$-module. If N is a central factor of G, V_N is homogeneous.

Proof. Let U be an irreducible submodule of V_N. Since $N.C_G(N) = G$, it follows from Clifford's Theorem that

$$V_N = \sum \{Ug : g \in C_G(N)\}.$$

Define the surjection $\alpha : U \to Ug$ by $u\alpha = ug$. Then for $u \in U$, $n \in N$,

$$\begin{aligned}(un)\alpha &= (un)g = u(ng),\\ &= u(gn) = (ug)n,\\ &= (u\alpha)n.\end{aligned}$$

Since α has an inverse $\beta : Ug \to U$ given by $v\beta = vg^{-1}$, it follows that α is an $\underline{E}N$-isomorphism. //

2.3.4 LEMMA Let the minimal normal subgroup M of G be abelian of exponent p, and consider M as an irreducible $\underline{F}(p)G$-module (where G acts on M by conjugation). If N is a normal subgroup of G containing M, and N is supplemented in G by a Sylow p-subgroup P of G, then

(i) the number of homogeneous components of M_N is a power of p, say p^c;

(ii) c is less than the class of P.

Proof. We shall use Clifford's Theorem. Let $M_1, \ldots, M_k$ be the homogeneous components of M_N, and let H_1 be the inertia group of M_1. Then $|G : H_1| = k$, and $N \leq H_1$. Since $NP = G$, it follows that $|G : H_1|$ is a power of p. This proves the first part. The elements of N stabilise each M_j, and so P must act transitively on $\{M_1, \ldots, M_{p^c}\}$. Since P contains M, it follows from Theorem 2.2.9 that the class of P is at least $c + 1$. //

We consider next the question of extending the field of scalars of a representation. Let G be a group, $\underline{E}$ a field, and $\underline{F}$ an extension field of $\underline{E}$. Suppose that V is an $\underline{E}G$-module, and that relative to an $\underline{E}$-basis $\{v_1, \ldots, v_k\}$, V affords a representation τ of G. Consider $\underline{F}$ as an $\underline{E}$-space, and form the $\underline{F}$-space $\underline{F} \otimes_{\underline{E}} V$, with basis $\{1 \otimes v_1, \ldots, 1 \otimes v_k\}$. We turn $\underline{F} \otimes_{\underline{E}} V$ into an $\underline{F}G$-module, denoted by $V^{\underline{F}}$, by defining for $g \in G$,

$$(1 \otimes v_i)g = 1 \otimes v_i g.$$

The representation of G over $\underline{F}$ afforded by $V^{\underline{F}}$ relative to $\{1 \otimes v_1, \ldots, 1 \otimes v_k\}$ is denoted by $\tau^{\underline{F}}$. An $\underline{E}G$-module V is said to be <u>absolutely irreducible</u> if $V^{\underline{F}}$ is irreducible for every extension field $\underline{F}$ of $\underline{E}$, and $\underline{E}$ is a <u>splitting field</u> for G if every irreducible $\underline{E}G$-module is absolutely irreducible. If $\underline{E}$ is a splitting field for G, and $\underline{F} \geq \underline{E}$, then $\underline{F}$ is also a splitting field for G [9, v, 11.3].

2.3.5 THEOREM Let $\underline{E}$ be a field, and let G be a group of exponent m. If $\underline{F}$ is the extension field of $\underline{E}$ obtained by adjunction of all the m^{th} roots of 1, then $\underline{F}$ is a splitting field for G.

Proof. Let $\underline{E}_0$ be the prime field of $\underline{E}$, and let $\underline{F}_0$ be the field obtained from $\underline{E}_0$ by adjunction of all the m^{th} roots of 1. Then $\underline{F}_0$ is a splitting field for G ([4, 41.1] if $\underline{E}_0 = \underline{Q}$, and [4, 70.24] if $\underline{E}_0 = \underline{F}(p)$ for some prime p). But $\underline{F}$ contains $\underline{F}_0$, so the Theorem is proved. //

We recall that a field $\underline{E}$ is called perfect in case every finite extension of $\underline{E}$ is separable over $\underline{E}$. Thus in particular, $\underline{E}$ is perfect if either $\underline{E}$ has characteristic zero, or $\underline{E}$ is a finite field [24, pages 64-65]. Representations $\sigma, \tau : G \to GL(n, \underline{E})$ of G are called Galois conjugate if there is a (field) automorphism α of $\underline{E}$ such that for all $g \in G$, $g\tau$ is obtained by applying α to the entries of $g\sigma$.

2.3.6 THEOREM [4, 70.15] Let G be a group, $\underline{E}$ a perfect field, and τ an irreducible representation of G over $\underline{E}$. If $\underline{F}$ is a splitting field for G which is a finite normal extension of $\underline{E}$, then $\tau^{\underline{F}}$ is completely reducible, and the irreducible components of $\tau^{\underline{F}}$ are all Galois conjugate. In particular, if τ is also faithful, then all the irreducible components of $\tau^{\underline{F}}$ are faithful. //

We shall need special cases of the following well-known lemma several times in this thesis.

2.3.7 LEMMA Let V and W be groups, $\alpha : V \to W$ an isomorphism, and suppose that X and Y are subgroups of AutV and AutW respectively. If $\alpha^{-1}X\alpha$ is conjugate to Y in AutW, the split-extensions of V by X and W by Y are isomorphic.

Proof. Recall that the split-extension VX of V by X is $\{(v,\mu) : v \in V, \mu \in X\}$, with group operation defined by

$$(u,\lambda).(v,\mu) = (u.v\lambda^{-1},\lambda\mu);$$

WY is defined similarly. Choose $\beta \in \mathrm{Aut}W$ such that $\beta^{-1}(\alpha^{-1}X\alpha)\beta = Y$, and denote $\alpha\beta$ by γ. Define $\psi : VX \to WY$ by

$$(v,\mu)\psi = (v\gamma,\gamma^{-1}\mu\gamma).$$

Now ψ is a homomorphism, for

$$\begin{aligned}(u,\lambda)\psi.(v,\mu)\psi &= (u\gamma,\gamma^{-1}\lambda\gamma).(v\gamma,\gamma^{-1}\mu\gamma),\\ &= (u\gamma.v\lambda^{-1}\gamma,\gamma^{-1}\lambda\mu\gamma),\\ &= ((u.v\lambda^{-1})\gamma,\gamma^{-1}\lambda\mu\gamma), \text{ since } \gamma \text{ is a homomorphism}\\ &= (u.v\lambda^{-1},\lambda\mu)\psi\\ &= ((u,\lambda)(v,\mu))\psi.\end{aligned}$$

The proof that ψ is bijective is easy, and will be omitted.//

We remark that under the additional hypotheses V abelian with $|V|$ and $|X|$ relatively prime (both of which will always be satisfied in our applications), the conjugacy of $\alpha^{-1}X\alpha$ and Y in AutW is a necessary as well as a sufficient condition for VX and WY to be isomorphic. Since this result is not needed in this thesis, no proof is offered.

One might ask why Lemma 2.3.7 is in this section at all, when it looks as if it ought to belong in section 2.2. In order to clarify this point, we first must recall the basic equivalence relation between linear groups; namely: linear groups X and Y operating on vector spaces V and W respectively (over some field) are called <u>linearly isomorphic</u> in case there is a bijective linear transformation $\gamma : V \to W$ such that $\gamma^{-1}X\gamma = Y$. Thus if V is elementary abelian, say of exponent p, it can be thought of as a vector space over $\underline{F}(p)$, in which case X is a linear group operating on V. In this case Lemma 2.3.7 (together with the above remark) states that VX and WY are isomorphic if and only if X and Y are linearly isomorphic. In fact we apply Lemma 2.3.7 typically in the following situation (see the work of Higman in section 2.4 and the proof of Theorem 4.3.1. We are given faithful $\underline{E}G$-modules M and N, and we want to try to decide when the split-extensions MG and NG are isomorphic. We have embeddings $\mu : G \to GL(M)$ and $\nu : G \to GL(N)$ given by

$$m(g\mu) = mg, \quad n(g\nu) = ng.$$

Lemma 2.3.7 tells us that MG is isomorphic to NG whenever $G\mu$ and $G\nu$ are linearly isomorphic. We shall follow John Cossey [3, page 46], in that we say that $\underline{E}G$-modules M and N are <u>linearly isomorphic</u> in case $G\mu$ and $G\nu$ are linearly isomorphic. Whereas this terminology is in many respects natural, there is one in which it is rather unfortunate. For, contrary to the normal effect of a qualifying adjective, "linear isomorphism" is a coarser relation between $\underline{E}G$-modules than is "isomorphism" (that is, isomorphic $\underline{E}G$-modules are always linearly isomorphic; the converse is false).

Of course, one often deals with matrix groups rather than with groups of linear transformations. Thus subgroups X and Y of

$GL(n,E)$ are said to be <u>linearly isomorphic</u> if, given an n-dimensional vector space V over $\underline{E}$ equipped with a basis, the linear groups corresponding to the natural action of X and Y on V are linearly isomorphic. In fact it is a trivial matter to show that X and Y are linearly isomorphic if and only if they are conjugate in $GL(n,\underline{E})$. Consequently, we shall call representations $\sigma,\tau : G \to GL(n,\underline{E})$ of G <u>linearly equivalent</u> in case $G\sigma$ and $G\tau$ are linearly isomorphic.

2.3.8 THEOREM [2, 2.5] Let G be a group, $\underline{E}$ a perfect field, and σ and τ faithful irreducible representations of G over $\underline{E}$. Then σ and τ are linearly equivalent if and only if $\sigma^{\underline{E^*}}$ has a composition factor linearly equivalent to a composition factor of $\tau^{\underline{E^*}}$, where $\underline{E^*}$ is the algebraic closure of $\underline{E}$.

Proof. The "only if" part of the Theorem is trivial. Let $\underline{F}$ be the extension field of $\underline{E}$ obtained by adjunction of all the m^{th} roots of 1, where m is the exponent of G. By Theorem 2.3.5, $\underline{F}$ is a splitting field for G. It is moreover a finite normal extension of $\underline{E}$, and hence by Theorem 2.3.6 both $\sigma^{\underline{F}}$ and $\tau^{\underline{F}}$ are completely reducible. Since $\underline{F}$ is a splitting field for G, and $\sigma^{\underline{E^*}} = (\sigma^{\underline{F}})^{\underline{E^*}}$, it follows that $\sigma^{\underline{E^*}}$ and $\tau^{\underline{E^*}}$ are completely reducible. Moreover the irreducible components of $\sigma^{\underline{E}^*}$ and $\tau^{\underline{E}^*}$ are all faithful. Suppose that $\sigma^{\underline{E}^*}$ has an irreducible component σ_1 of degree n linearly equivalent to an irreducible component τ_1 of $\tau^{\underline{E}^*}$. Then there is an element $x \in GL(n,\underline{E}^*)$ such that $G\tau_1 = x^{-1}(G\sigma_1)x$. Define the automorphism α of G by putting $g\alpha$ equal to the unique element of G such that $x^{-1}(g\sigma_1)x = (g\alpha)\tau_1$. Consider the faithful irreducible representation $\alpha^{-1}\sigma$ of G over $\underline{E}$; this is linearly equivalent to σ, for $G(\alpha^{-1}\sigma) = (G\alpha^{-1})\sigma = G\sigma$. Thus it is sufficient to prove that $\alpha^{-1}\sigma$ and τ are linearly equivalent. If $g \in G$,

$$
\begin{aligned}
g(\alpha^{-1}\sigma)^{\underline{E}^*} &= (g\alpha^{-1})\sigma^{\underline{E}^*} \\
&= g\alpha^{-1}(\sigma_1 \oplus \cdots \oplus \sigma_k), \quad \text{(say)} \\
&= g(\alpha^{-1}\sigma_1 \oplus \cdots \oplus \alpha^{-1}\sigma_k).
\end{aligned}
$$

Now $x^{-1}(g\alpha^{-1}\sigma_1)x = g\tau_1$, and so $\alpha^{-1}\sigma_1$ is equivalent to τ_1. Thus $\alpha^{-1}\sigma$ and τ are irreducible representations of G such that $(\alpha^{-1}\sigma)^{\underline{E}^*}$ and $\tau^{\underline{E}^*}$ are completely reducible and have a composition factor τ_1 in common. It follows from [4, 29.6] that $\alpha^{-1}\sigma$ and τ are equivalent, and hence that σ and τ are linearly equivalent. //

Various forms of the "outer tensor product theorem" (Theorem 2.3.9 below) have appeared in print (see for example [9, v, 10.3]). However, to the best of my knowledge, the version I wish to quote has not. It occurs as Theorem 1.3.15 in John Cossey's thesis [3], with a proof by L.G. Kovács.

Let A and B be groups, $G = A \times B$ their direct product, and $\underline{E}$ a field. If U and V are respectively $\underline{E}A$- and $\underline{E}B$-modules, we define the "outer tensor product" $U \#_{\underline{E}} V$ of U and V to be the $\underline{E}G$-module whose underlying $\underline{E}$-space is $U \otimes_{\underline{E}} V$, with module operation defined by

$$(u \otimes v)ab = ua \otimes vb$$

(obvious notation), and extended by linearity to all of $\underline{E}G$ and $U \otimes_{\underline{E}} V$. If σ, τ are representations afforded by U and V relative to $\underline{E}$-bases $\{u_1, \ldots, u_k\}$, $\{v_1, \ldots, v_\ell\}$, then that afforded by $U \#_{\underline{E}} V$ relative to $\{u_i \otimes v_j : 1 \leq i \leq k,\ 1 \leq j \leq \ell\}$ is denoted by $\sigma \#_{\underline{E}} \tau$. Henceforth, if the context allows it, we drop the subscript on $\otimes$ and $\#$.

2.3.9 THEOREM Let A and B be groups, $G = A \times B$ their direct product, and $\underline{E}$ a field.

(i) If V is an irreducible $\underline{E}G$-module, and if an irreducible submodule V_1 of V_A is absolutely irreducible, then V is isomorphic to $V_1 \# V_2$, where V_2 is an irreducible submodule of V_B. (We remark that any doubts concerning grammar in statement (i) ought to be dispelled after observing that by Lemma 2.3.3, V_A and V_B are homogeneous.)

(ii) If U is an absolutely irreducible $\underline{E}A$-module, and V is an irreducible $\underline{E}B$-module, then $U \# V$ is an irreducible $\underline{E}G$-module. //

2.3.10 COROLLARY The conclusions of Theorem 2.3.9 are still valid when G is replaced by a central product of A and B.

Proof. (i) Let H be a central product of A and B; then there is a normal subgroup, say N, of $Z(A) \times Z(B)$ such that $H \cong G/N$. Let $\alpha : G \to H$ be an epimorphism, and suppose that V is an irreducible $\underline{E}H$-module, where $\underline{E}$ is some field. We can turn V into an $\underline{E}G$-module, which we denote by $\overline{V}$, by defining $vg = v(g\alpha)$. Since α is surjective, $\overline{V}$ is irreducible. By Lemma 2.3.3, V_A and V_B are homogeneous; let V_1 and V_2 be irreducible submodules of V_A, V_B respectively. Now $\overline{V}_1$ is absolutely irreducible if and only if V_1 is. It follows from Theorem 2.3.9 (i) that if V_1 is absolutely irreducible, $\overline{V} \cong \overline{V}_1 \# \overline{V}_2$. But $\overline{V}_1 \# \overline{V}_2 = \overline{V_1 \# V_2}$, and so, since α is surjective, $V \cong V_1 \# V_2$, as required.

(ii) Suppose U is an absolutely irreducible $\underline{E}A$-module and V is an irreducible $\underline{E}B$-module, and form $U \# V$. Since $\overline{U\# V} = \overline{U} \# \overline{V}$, it follows from Theorem 2.3.9 (ii) that $\overline{U\# V}$ is irreducible. Hence $U \# V$ is irreducible. //

2.3.11 COROLLARY (c.f. [2, 2.7]) Let A and B be groups, and let $\underline{E}$ be a splitting field for A. Suppose that A has the property that every pair of faithful irreducible $\underline{E}$-representations of it are linearly equivalent. If B has the same property, and C is a central product of A and B, then C has the same property. (This includes the case where A and B have coprime orders and $C = A \times B$.)

Proof. Let σ and τ be faithful irreducible representations of C over $\underline{E}$. By Corollary 2.3.10, σ is equivalent to $\sigma_1 \# \sigma_2$, and τ is equivalent to $\tau_1 \# \tau_2$, where $\sigma_1, \sigma_2, \tau_1$ and τ_2 are irreducible components of $\sigma_A, \sigma_B, \tau_A$ and τ_B respectively (since $\underline{E}$ is a splitting field for A). By Lemma 2.3.3, $\sigma_1, \sigma_2, \tau_1$ and τ_2 are all faithful. Hence there is $m \in \underline{P}$, such that $\sigma_1, \tau_1 : A \to GL(m,\underline{E})$, and there is $x \in GL(m,\underline{E})$ such that $A\sigma_1{}^x = A\tau_1$. Similarly there is $n \in \underline{P}$ and $y \in GL(n,\underline{E})$ such that $B\sigma_2{}^y = B\tau_2 \leq GL(n,\underline{E})$. Let $x \otimes y$ be the Kronecker product of x and y [4, page 69]; then for all $a,b \in C$,

$$\begin{aligned}(ab(\sigma_1 \# \sigma_2))^{x \otimes y} &= (a\sigma_1 \otimes b\sigma_2)^{x \otimes y}, \\ &= (a\sigma_1)^x \otimes (b\sigma_2)^y \\ &= a_1\tau_1 \otimes b_1\tau_2, \text{ for some } a_1 \in A, b_1 \in B \\ &= a_1b_1(\tau_1 \# \tau_2) .\end{aligned}$$

Hence $\sigma_1 \# \sigma_2$ is linearly equivalent to $\tau_1 \# \tau_2$, and so σ is linearly equivalent to τ. //

2.4 Varieties of groups

Throughout this thesis, "variety" means "variety of groups". From the definitions available to us, we choose that a variety is a class of (not necessarily finite) groups which is closed under the operations of taking subgroups, homomorphic images, and Cartesian products.

I have been rather fortunate in that Hanna Neumann's book [18] became generally available shortly before I began my stay at the A.N.U. Consequently, we can follow the examples set in sections 2.2 and 2.3, in that *it suffices for our purposes to take for granted sections 1 through 6 (pages 1-27) of Chapter 1 of [18].* We shall as far as possible adhere to the notation of [18] (but see also section 1.1 above).

Thus we are left to recall three definitions from the rest of [18]: if $\underline{\underline{U}}$ and $\underline{\underline{V}}$ are varieties, we define their product (variety) $\underline{\underline{UV}}$ [18, 21.11] to be the variety consisting of all groups that are extensions of a group in $\underline{\underline{U}}$ by a group in $\underline{\underline{V}}$. That this class does in fact constitute a variety is easily checked using the definition of variety given above. A (finite) group is called critical in case it does not belong to the variety generated by its proper sections [18, 51.31]. Finally, if M is a normal subgroup of A, and N is a normal subgroup of B, we say that M (qua normal subgroup of A) is similar to N (qua normal subgroup of B) if there are isomorphisms $\mu : M \to N$, $\nu : A/C_A(M) \to B/C_B(N)$, such that (with the obvious notation)

$$(m^{aC_A(M)})\mu = m\mu^{(aC_A(M))\nu}$$

We write $(M \trianglelefteq A) \sim (N \trianglelefteq B)$, or, if the context allows it, $M \sim N$ [18, 53.11].

2.4.1 LEMMA Let $\underline{\underline{V}}$ be a variety, G a group in $\underline{\underline{V}}$, and N an abelian normal subgroup of G. Then $\underline{\underline{V}}$ also contains a group G^*, which has a complemented, self-centralising (abelian) normal subgroup N^*, with $N \sim N^*$.

Proof. (L.G. Kovács) Denote the direct square $G \times G$ of G by K, and the normal subgroup $N \times E$ of K by M. Let L be the diagonal in K, and denote $L \cap (C \times C)$ by J, where C is $C_G(N)$. Put ML/J equal to G^*, MJ/J equal to N^*, and L/J equal to H^*. Then $G^* \in \underline{\underline{V}}$, and N^* is a self-centralising, abelian normal subgroup of G^*, which is complemented in G^* by H^*. Define $\mu : N \to N^*$ by $n\mu = (n,e)J$, and $\nu : G/C \to G^*/N^*$ by $gC\nu = ((g,g)J)N^*$; it is easy to see that μ and ν are in fact isomorphisms. Finally,

$$\begin{aligned}(n^{gC})\mu = n^g\mu &= (n^g,e)J \\ &= (n,e)^{(g,g)}J \\ &= ((n,e)J)^{((g,g)J)N^*} \\ &= n\mu^{gC\nu}. \end{aligned}$$

//

In the opening paragraph of Chapter 1 of this thesis, we gave a statement of the Oates-Powell Theorem. Kovács and Newman [13] have a somewhat different version of this same result which we now outline (since it is the more convenient for our purposes). For positive integers e, m and c, denote by $\underline{\underline{C}}(e,m,c)$ the class of all (not necessarily finite) groups of exponent dividing e whose chief-sections have order (at most) m, and whose nilpotent sections have class (at most) c. They prove

2.4.2 THEOREM For all positive integers e, m and c, $\underline{\underline{C}}(e,m,c)$ is a Cross variety. Furthermore, a variety $\underline{\underline{V}}$ is Cross if and only if there exist positive integers e, m and c such that $\underline{\underline{V}}$ is a subclass of $\underline{\underline{C}}(e,m,c)$. //

2.4.3 COROLLARY (i) Let $\underline{\underline{V}}$ be a jnC variety of finite exponent n in which the nilpotent groups do not form a subvariety. Then n is a prime-power. Hence if $\underline{\underline{V}}$ is also locally finite, then it is locally nilpotent.

(ii) Let $\underline{\underline{V}}$ be a locally finite non-Cross variety. If $\underline{\underline{V}}$ is locally nilpotent, then the nilpotent groups in $\underline{\underline{V}}$ do not form a subvariety.

Proof. (i) Since the nilpotent groups in $\underline{\underline{V}}$ do not form a subvariety, for all $c \in \underline{P}$ there is a (not necessarily finite) nilpotent group, say G_c, in $\underline{\underline{V}}$ with class precisely c. But $\underline{\underline{V}}$ is jnC, and so by Theorem 2.4.2, $\underline{\underline{V}}$ is $\text{var}\{G_c : c \in \underline{P}\}$. Let $p_1,\ldots,p_k$ be the primes dividing n, and let $G_{c,i}$ be the set consisting of those elements of G_c whose orders are powers of p_i. Since G_c is nilpotent of finite exponent, it is locally finite, and so if $g,h \in G_c, \langle g,h\rangle$ is finite and nilpotent. It follows that each $G_{c,i}$ is a subgroup of G_c, the exponent of $G_{c,i}$ is the p_i-share of the exponent of G_c, and

$$G_c = G_{c,1} \times \ldots \times G_{c,k} .$$

Thus there is an i, $1 \leq i \leq k$, such that the groups $\{G_{c,i} : c \in \underline{P}\}$ have unbounded class. Then as $\underline{\underline{V}}$ is jnC, it follows from Theorem 2.4.2 that

$$\underline{\underline{V}} = \operatorname{var}\{G_{c,i} : c \in \underline{P}\},$$

and so the exponent of $\underline{\underline{V}}$ is a power of p_i.

(ii) If the nilpotency class of the nilpotent groups in $\underline{\underline{V}}$ is at most c, then, since $\underline{\underline{V}}$ is locally finite and locally nilpotent, $\underline{\underline{V}}$ is a subvariety of $\underline{\underline{N}}_c$ [18, 15.61]. But then $\underline{\underline{V}}$ is generated by its free group of rank c [18, 35.12], which is a finite group. It follows from the Oates-Powell Theorem that $\underline{\underline{V}}$ is Cross, a contradiction. //

Graham Higman ([6] and [8, section 2.7]) has developed methods which are extremely useful in questions concerning varieties $\underline{\underline{V}}$ such that

$$\underline{\underline{B}} \leq \underline{\underline{V}} \leq \underline{\underline{A}}_m\underline{\underline{B}},$$

where $\underline{\underline{B}}$ is a localy finite variety of exponent prime to m.

Observe that such varieties $\mathbf{V}$ are locally finite [18, 21.14], and hence are generated by their (finite) monolithic groups [18, 51.32 and 51.41]. As we need to apply his results in the case $\mathbf{B} = \mathbf{N}_2 \wedge \mathbf{B}_n$ for the proof of Theorem 4.3.1 (see section 4.3), we conclude this chapter by giving an account of his ideas. Whereas we have aimed at brevity, we have nonetheless felt obliged to supply some detail, most of which was suppressed in the original.

In [6], Higman restricts his attention to the case where m is a prime, say $m = p$; we now make this assumption until further notice. Let G be a monolithic group in $\mathbf{A}_p\mathbf{B} - \mathbf{B}$, and denote $\mathbf{B}(G)$ by N. Since $N \varepsilon \mathbf{A}_p$ and $G/N \varepsilon \mathbf{B}$, the Schur-Zassenhaus Theorem [9, 1, 18.2] assures us that N is complemented in G, say by H. It follows that N is self-centralising, and Maschke's Theorem tells us that N is the monolith. Thus, in the language of group representations, H is faithfully and irreducibly represented (by conjugation) on N over $\underline{F}(p)$. In this way, G determines, and by Lemma 2.3.7 is determined up to isomorphism by, a linear isomorphism class $L(G)$ of irreducible linear groups in $\mathbf{B}$ over $\underline{F}(p)$ of dimension greater than zero. (We shall adopt the convention that a linear group always acts on a space of dimension greater than zero.) If $\mathbf{V}$ is a variety such that

$$\mathbf{B} \leq \mathbf{V} \leq \mathbf{A}_p\mathbf{B},$$

put $L(\underline{\underline{V}})$ equal to the union of the $L(G)$, where G runs through the monolithic groups in $\underline{\underline{V}} - \underline{\underline{B}}$; then the correspondence $\underline{\underline{V}} \to L(\underline{\underline{V}})$ is an (inclusion-preserving) injection from $\{\underline{\underline{V}} : \underline{\underline{B}} \leq \underline{\underline{V}} \leq \underline{\underline{A}}_p\underline{\underline{B}}\}$ to the set of classes of irreducible linear groups in $\underline{\underline{B}}$ over $\underline{F}(p)$. Hence $\underline{\underline{V}}$ is Cross if and only if $\underline{\underline{B}}$ is Cross and $L(\underline{\underline{V}})$ is the union of only finitely many linear isomorphism classes. Higman's main theorem [6, 4.6] is a characterisation of the classes of irreducible linear groups that can occur as $L(\underline{\underline{V}})$'s; in order to state it, we need the following definitions. If X and Y are linear groups over $\underline{F}(p)$ on spaces V, W respectively, we say that Y is a <u>linear factor</u> of X (and we write $Y \prec X$) in case X has a subgroup X_0, and V has an X_0-admissible subspace V_0, such that the restriction of X_0 to V_0 is linearly isomorphic to Y. A class $\underline{\underline{X}}$ of irreducible linear groups is called <u>closed</u> if every irreducible linear factor of every group in $\underline{\underline{X}}$ also belongs to $\underline{\underline{X}}$. Note that a closed class is a union of complete linear isomorphism classes.

2.4.4 THEOREM Let $\underline{\underline{X}}$ be a class of irreducible linear groups in $\underline{\underline{B}}$ over $\underline{F}(p)$. Then $\underline{\underline{X}}$ is closed if and only if there is a variety $\underline{\underline{V}}$, $\underline{\underline{B}} \leq \underline{\underline{V}} \leq \underline{\underline{A}}_p\underline{\underline{B}}$, such that $\underline{\underline{X}} = L(\underline{\underline{V}})$. Thus L gives an inclusion-preserving bijection from $\{\underline{\underline{V}} : \underline{\underline{B}} \leq \underline{\underline{V}} \leq \underline{\underline{A}}_p\underline{\underline{B}}\}$ to the set of closed classes of irreducible linear groups in $\underline{\underline{B}}$ over $\underline{F}(p)$. //

Higman applies Theorem 2.4.4 to the case $\underline{\underline{B}} = \underline{\underline{T}}_q$: let C_n be a central product with cyclic centre of n copies of Q_q (see section 1.1), and let X_n be an irreducible linear group over $\underline{F}(p)$ which is isomorphic (as abstract group) to C_n. (Such an X_n exists, otherwise $M(C_n)$ would be contained in the kernel of every irreducible $\underline{F}(p)$-representation of C_n, and hence would be contained in the kernel of the regular representation of C_n over $\underline{F}(p)$; which is impossible.) If $\underline{\underline{Y}}_0 = \{X_n : n \in \underline{P}\}$, and $\underline{\underline{Y}}$ is the closure of (i.e. least closed class containing) $\underline{\underline{Y}}_0$, then ([6, 4.10]):

2.4.5 THEOREM $\underline{\underline{Y}} = L(\underline{\underline{A}}_p\underline{\underline{T}}_q)$, and $\underline{\underline{A}}_p\underline{\underline{T}}_q$ is jnC. //

In [8, section 2.7], Higman observes that for certain problems it is unnecessary to restrict m to being a prime. For suppose that m is not prime, and let $m = k\ell$, where k and ℓ are coprime. Then

$$\underline{\underline{V}} = (\underline{\underline{V}} \wedge \underline{\underline{A}}_k\underline{\underline{B}}) \vee (\underline{\underline{V}} \wedge \underline{\underline{A}}_\ell\underline{\underline{B}}),$$

for all subvarieties $\underline{\underline{V}}$ of $\underline{\underline{A}}_m\underline{\underline{B}}$. Thus all investigations of the subvarieties of $\underline{\underline{A}}_m\underline{\underline{B}}$ are effectively reduced to investigations of subvarieties of $\underline{\underline{A}}_{p^a}\underline{\underline{B}}$, for prime divisors p^a of m. In particular, if $\underline{\underline{V}}$ is a jnC subvariety of $\underline{\underline{A}}_m\underline{\underline{B}}$, then $\underline{\underline{V}}$ lies in some $\underline{\underline{A}}_{p^a}\underline{\underline{B}}$ (as finite joins of Cross varieties are Cross, on account of the Oates-Powell Theorem).

Let us therefore consider subvarieties of $\underline{\underline{A}}_{p^a}\underline{\underline{B}}$ containing $\underline{\underline{B}}$. Suppose X is an irreducible linear group in $\underline{\underline{B}}$ over $\underline{F}(p)$ of dimension d. For each b, $1 \leq b \leq a$, let $P(b)$ be the free group of rank d of $\underline{\underline{A}}_{p^b}$ (i.e. $P(b)$ is homocyclic of exponent p^b and order p^{bd}), freely generated by $x_1,\ldots,x_d$. If $\alpha : P(b) \to P(b)/D(P(b))$ is the natural epimorphism, we define the homomorphism $\beta : \mathrm{Aut}P(b) \to \mathrm{Aut}[P(b)/D(P(b))]$ by

$$(x\alpha)\lambda\beta = (x\lambda)\alpha,$$

where $\lambda \in \mathrm{Aut}P(b)$. In fact, β is a surjection. For, as $\{x_1,\ldots,x_d\}$ generates $P(b)$ freely, to each η in $\mathrm{Aut}[P(b)/D(P(b))]$ one can define an endomorphism ξ of $P(b)$ by choosing $x_i\xi$ arbitrarily in $x_i\alpha\eta$: that is, so that $\alpha\eta = \xi\alpha$. Since $\alpha\eta$ is an epimorphism, $P(b)\xi$ must supplement the kernel of α in $P(b)$; since the kernel of α is $D(P(b))$, ξ must be surjective. But $P(b)$ is finite, and so ξ is an automorphism. It follows now that $\xi\beta = \eta$, proving the surjectivity of β.

Suppose for convenience that X acts on $P(b)/D(P(b))$; that is,

$X \leq \mathrm{Aut}[P(b)/D(P(b))]$.

Let Y be the preimage of X under β; then $Y \geq \ker\beta$ and $Y/\ker\beta \cong X \in \underline{\underline{B}}$. Now $\ker\beta$ is a p-group [9,iii, 3.18], so that by the Schur-Zassenhaus Theorem [9, 1, 18.2], $\ker\beta$ is complemented in Y, all complements being conjugate in Y (and hence in $\mathrm{Aut}P(b)$). Thus if $X(b)$ is the split-extension of $P(b)$ by a complement of $\ker\beta$ in Y, $X(b)$ is determined up to isomorphism (Lemma 2.3.7), while b and (the linear isomorphism type of) X are isomorphism invariants of $X(b)$. Moreover, $X(b)$ is monolithic (the monolith of $X(b)$ being the socle of $P(b)$), and hence critical [16, 1.6.6]. Also, if $1 \leq c \leq a$, and Y is an irreducible linear group in $\underline{\underline{B}}$ over $\underline{F}(p)$,

$$Y(c) \in \mathrm{var}X(b) \iff c \leq b \text{ and } Y \prec X.$$

If $\underline{\underline{B}} \leq \underline{\underline{V}} \leq \underline{\underline{A}}_{p^a}\underline{\underline{B}}$, put, for $1 \leq b \leq a$,

$$L_b(\underline{\underline{V}}) = \{X : X \text{ irreducible linear group in } \underline{\underline{B}} \text{ over } \underline{F}(p) \text{ such that } X(b) \in \underline{\underline{V}}\}.$$

Then

$$L_1(\underline{\underline{V}}) \supseteq \dots \supseteq L_a(\underline{\underline{V}}),$$

and

$$\underline{\underline{V}} = \underline{\underline{B}} \vee \mathrm{var}\{X(1) : X \in L_1(\underline{\underline{V}})\} \vee \dots \vee \mathrm{var}\{X(a) : X \in L_a(\underline{\underline{V}})\}.$$

Also, if $\underline{\underline{U}}$ is another such variety,

$$\underline{\underline{U}} \leq \underline{\underline{V}} \iff L_b(\underline{\underline{U}}) \subseteq L_b(\underline{\underline{V}}), \quad 1 \leq b \leq a.$$

Moreover, if $\underline{\underline{X}}_1, \ldots, \underline{\underline{X}}_a$ are closed classes of irreducible linear groups in $\underline{\underline{B}}$ over $\underline{F}(p)$ such that $\underline{\underline{X}}_1 \supseteq \ldots \supseteq \underline{\underline{X}}_a$, then the variety $\underline{\underline{V}}$ defined by

$$\underline{\underline{V}} = \underline{\underline{B}} \vee \mathrm{var}\{X(1) : X \in \underline{\underline{X}}_1\} \vee \ldots \vee \mathrm{var}\{X(a) : X \in \underline{\underline{X}}_a\}$$

is the unique variety such that $\underline{\underline{B}} \leq \underline{\underline{V}} \leq \underline{\underline{A}}_p{}^a\underline{\underline{B}}$ and $L_b(\underline{\underline{V}}) = \underline{\underline{X}}_b$, $1 \leq b \leq a$. Thus there is a bijection between the subvarieties of $\underline{\underline{A}}_p{}^a\underline{\underline{B}}$ containing $\underline{\underline{B}}$ and decreasing sequences of length a of closed classes of irreducible linear groups in $\underline{\underline{B}}$ over $\underline{F}(p)$. In particular, one can deduce that such a variety $\underline{\underline{V}}$ is non-Cross if and only if either $\underline{\underline{B}}$ is non-Cross or $L_1(\underline{\underline{V}})$ is the union of infinitely many linear isomorphism classes. Hence if $\underline{\underline{B}}$ is Cross, but $\underline{\underline{V}}$ is not, the jnC subvarieties of $\underline{\underline{V}}$ all lie in $\underline{\underline{A}}_p\underline{\underline{B}}$.

We shall need the following consequence of the preceding discussion

2.4.6 THEOREM If $\underline{\underline{B}}$ is a Cross variety, and $\underline{\underline{V}}$ is a jnC subvariety of $\underline{\underline{A}}_m\underline{\underline{B}}$, where the exponent of $\underline{\underline{B}}$ is prime to m, then there is a prime, say p, dividing m, such that $\underline{\underline{V}}$ is a subvariety of $\underline{\underline{A}}_p\underline{\underline{B}}$. //

CHAPTER 3

REDUCIBLE JNC VARIETIES

In this chapter we shall prove that a jnC variety is reducible if and only if it is soluble of finite exponent. The proof of this Theorem occupies the whole of section 3.2. In section 3.1, we deduce some preparatory results, the most substantial of which is Theorem 3.1.1. This arose from attempts to generalise the proof of Lemma 5 of [21]. I am indebted to L.G. Kovács for suggesting it to me.

3.1 Some preparatory results

The statement of Theorem 3.1.1 which we give here serves also to introduce some notation.

3.1.1 THEOREM Let $\underline{\underline{V}}$ be a variety of finite exponent n in which the nilpotent groups have class (at most) c, and let B be a nonabelian (finite) simple group. Suppose that $\underline{\underline{V}}$ contains an infinite set Γ of pairwise-nonisomorphic (finite) monolithic groups, such that the monolith $M(G)$ of each group G in Γ is isomorphic to a direct power, say $B^{\alpha(G)}$, of B. (In this way we define a function $\alpha : \Gamma \to \underline{P}$.) Then $\underline{\underline{V}}$ is non-Cross, and it has a non-Cross subvariety to which B does not belong.

The proof of Theorem 3.1.1 falls naturally into three steps, the first two of which we isolate as lemmas. First, the claim that $\underline{\underline{V}}$ is non-Cross is easily established:

3.1.2 LEMMA $\alpha(\Gamma)$ is an infinite subset of $\underline{P}$, and so $\text{var}\Gamma$ is a non-Cross subvariety of $\underline{\underline{V}}$. In particular, $\underline{\underline{V}}$ is non-Cross.

Proof. Suppose to the contrary that $\alpha(\Gamma)$ is a finite subset of $\underline{P}$, say $\alpha(G) < a$ for all $G \in \Gamma$. Then $\{|M(G)| : G \in \Gamma\}$ is bounded by $|B|^a$. Now $G/C_G(M(G))$ is isomorphic to a subgroup of $\text{Aut}M(G)$, and $C_G(M(G))$ is trivial for $G \in \Gamma$. Hence $\{|G| : G \in \Gamma\}$ is bounded by $(|B|^a)!$, and so Γ is a finite set. This contradiction establishes the first claim of the Lemma; all the other claims follow from it and Theorem 2.4.2. //

3.1.3 LEMMA There is a prime p and an infinite set Δ of monolithic groups in $\underline{\underline{V}}$, such that

(i) $\text{var}\Delta$ is a non-Cross subvariety of $\underline{\underline{V}}$;

(ii) the monolith of each group H in Δ is isomorphic to $B^{p^{\beta(H)}}$, and $\beta(\Delta)$ is an infinite subset of $\underline{P}$;

(iii) if $H \in \Delta$, $M(H)$ is supplemented in H by a Sylow p-subgroup.

Proof. Let $G \varepsilon \Gamma$, and suppose that the direct factors of $M(G)$ are $B_1, \ldots, B_{\alpha(G)}$. Denote $N_G(B_i)$ by N_i, and $\cap\{N_i : 1 \leq i \leq \alpha(G)\}$ by N. By Corollary 2.2.6, G is represented (by conjugation) as a transitive permutation group on $\{B_1, \ldots, B_{\alpha(G)}\}$ with kernel N; the stabiliser of the "point" B_i being N_i/N. It follows from Lemma 2.2.7 (i) that $|G : N_i| = \alpha(G)$, $1 \leq i \leq \alpha(G)$, and so the prime divisors of $\alpha(G)$ all divide n. But n is finite, and by Lemma 3.1.2, $\alpha(\Gamma)$ is an infinite subset of $\underline{P}$; hence there is a prime, say p, such that $\alpha_p(\Gamma)$ is an infinite subset of $\underline{P}$, where $\alpha_p(G)$ is the p-share of $\alpha(G)$. Let P be a Sylow p-subgroup of G; then by Lemma 2.2.7, the orbits of PN/N have cardinality a power, say $p^{\beta(G)}$, of p, and $p^{\beta(G)} \geq \alpha_p(G)$. Denote $p^{\beta(G)}$ by $\gamma(G)$, and suppose that the direct factors of $M(G)$ have been numbered so that the orbit of PN/N containing B_1 is $\{B_1, \ldots, B_{\gamma(G)}\}$. Put $\langle B_1, P\rangle$ equal to A, and choose $\Delta = \{A/Z_\infty(A) : G \varepsilon \Gamma\}$.

If K is the normal closure of B_1 in A

$$K = B_1 \times \ldots \times B_{\gamma(G)}.$$

By Lemma 2.2.5, K is a minimal normal subgroup of A; but A need not be monolithic, as there may be (necessarily central) minimal normal subgroups of A contained in P. Thus $A/Z_\infty(A)$

is monolithic, and since $Z_\infty(A)$ avoids K, the monolith of $A/Z_\infty(A)$ is isomorphic to K. An application of Theorem 2.4.2 completes the proof. //

3.1.4 COROLLARY B is a p'-group.

Proof. Since $\beta(\Delta)$ is an infinite subset of $\underline{P}$, there is a group, say G_1, in Δ with $\beta(G_1) > c$. Suppose that P_1 is a Sylow p-subgroup of G_1, and let B_1 be a direct factor of $M(G_1)$. If p divided $|B|$, P_1 would intersect $M(G_1)$ nontrivially, and so $P_1 \cap B_1$ would be nontrivial. But then Corollary 2.2.10 would imply that P_1 has class greater than c, which would be a contradiction. //

We are now ready to prove that $\underline{\underline{V}}$ has a non-Cross subvariety to which B does not belong.

Let $G \varepsilon \Delta$, and let P be a Sylow p-subgroup of G; then by Lemma 3.1.3 and Corollary 3.1.4, G is a split-extension of $M(G)$ by P. Denote the direct factors of $M(G)$ by $B_1,\ldots,B_{\gamma(G)}$ (where as before $\gamma(G) = p^{\beta(G)}$), $N_G(B_i)$ by N_i, and $\cap\{N_i : 1 \leq i \leq \gamma(G)\}$ by N. By Lemmas 2.2.3 and 2.2.4, B_1 has a non-nilpotent, proper, intravariant subgroup, say T_1, such that $N_P(T_1)$ contains (and hence equals) $P \cap N_1$. Denote $\langle T_1, P\rangle$ by H, and the normal closure of T_1 in H by T. If $T \cap B_i$ is T_i,

$$T = T_1 \times \ldots \times T_{\gamma(G)}.$$

Suppose that $Z_\infty(T)$ is Y and $Z_\infty(T_j)$ is Y_j; then

$$Y = Y_1 \times \ldots \times Y_{\gamma(G)},$$

and since T_i is non-nilpotent, $1 \leq i \leq \gamma(G)$,

$$Y_i < T_i.$$

Observe that Y is normal in H, being characteristic in T. Denote H/Y by $\overline{H}$, T/Y by $\overline{T}$, T_iY/Y by $\overline{T}_i$, PY/Y by $\overline{P}$ and $\{\overline{H} : H \in \Delta\}$ by Λ. Then $\overline{H}$ is a split-extension of $\overline{T}$ by $\overline{P}$, and

$$\overline{T} = \overline{T}_1 \times \ldots \times \overline{T}_{\gamma(G)}.$$

Since $Z(\overline{T}_i)$ is trivial, and $\overline{P}$ connects $\{\overline{T}_i : 1 \leq i \leq \gamma(G)\}$ transitively, Lemma 2.2.5 implies that a minimal normal subgroup $\overline{L}$ of $\overline{H}$ contained in $\overline{T}$ intersects each $\overline{T}_i$ nontrivially. Thus $|\overline{L}| \geq \gamma(G) = p^{\beta(G)}$, and so by Theorem 2.4.2, $\mathrm{var}\Lambda$ is a non-Cross subvariety of $\underline{\underline{V}}$. Observe that $\mathrm{var}\Lambda$ is a subvariety of $(\mathrm{var}T_1).\underline{\underline{N}}$, where

$\underline{\underline{N}}$ is the variety of nilpotent groups in $\underline{\underline{V}}$. Since B is critical [18, 51.34], it does not belong to $\mathrm{var}T_1$. Hence B does not belong to $\mathrm{var}\Lambda$, and the proof of Theorem 3.1.1 is complete. //

3.1.5 COROLLARY Let $\underline{\underline{V}}$ be a jnC variety of finite exponent in which the nilpotent groups form a subvariety, and let B be a nonabelian simple group. Then $\underline{\underline{V}}$ contains only finitely many (isomorphism classes of) monolithic groups whose monoliths have a direct factor isomorphic to B. //

We conclude this section with two lemmas which describe some important properties of reducible jnC varieties.

3.1.6 LEMMA (i) A reducible jnC variety is locally finite, and contains only finitely many (isomorphism classes of) finite simple groups.

(ii) A jnC variety is reducible and locally nilpotent if and only if it is $\underline{\underline{A}}_p\underline{\underline{A}}_p$ for some prime p.

Proof (i) Suppose that $\underline{\underline{V}}$ is a reducible jnC variety, say $\underline{\underline{V}}$ is a subvariety of $\underline{\underline{V}}_1\underline{\underline{V}}_2$, where the $\underline{\underline{V}}_i$ are proper (and hence Cross) subvarieties of $\underline{\underline{V}}$. Since Cross varieties are locally finite, the first part of (i) is an immediate consequence of [18, 21.14]. For the second part of (i), observe that a simple group in $\underline{\underline{V}}$ belongs either to $\underline{\underline{V}}_1$ or to $\underline{\underline{V}}_2$. But simple groups are critical [18, 51.34], and Cross varieties contain only finitely many (isomorphism classes of) critical groups.

(ii) If $\underline{\underline{V}}$ is also locally nilpotent, the Oates-Powell Theorem shows that both $\underline{\underline{V}}_1$ and $\underline{\underline{V}}_2$ are nilpotent, and hence that $\underline{\underline{V}}$ is soluble and locally nilpotent. It then follows from [15, Theorem 5] that $\underline{\underline{V}}$ is $\underline{\underline{A}}_p\underline{\underline{A}}_p$ for some prime p. The "if" part of (ii) is trivial. //

In section 6.1, we shall prove a partial converse to part (i) of Lemma 3.1.6; namely, if a locally finite jnC variety is not locally nilpotent and contains only finitely many (isomorphism classes of) finite simple groups, then it is reducible.

We state the final lemma of this section in its fullest generality; this requires us to use Corollary 3.1.5 in its proof. We observe that if $\underline{\underline{V}}$ is soluble, this is no longer necessary, and so the results of Chapters 4 and 5 are independent of Corollary 3.1.5.

3.1.7 LEMMA Let $\underline{\underline{V}}$ be a locally finite jnC variety which contains only finitely many (isomorphism classes of) finite simple groups. If $\underline{\underline{V}}$ is not locally nilpotent, there is a prime p and a (countably) infinite set Γ of monolithic groups in $\underline{\underline{V}}$, such that:

(i) $\underline{\underline{V}} = \mathrm{var}\Gamma$;

(ii) the monolith of each group in Γ is complemented, self-centralising and has exponent p;

(iii) the orders of the monoliths of the groups in Γ form an infinite set.

In particular, the conclusions follow when $\underline{\underline{V}}$ is a reducible jnC variety.

Proof. By Corollary 2.4.3 (i), there is a bound on the nilpotency class of the nilpotent groups in $\underline{\underline{V}}$. Since a locally finite variety has finite exponent, and is generated by its finite groups [18, 15.61], Theorem 2.4.2 implies that the orders of the chief-sections of the finite groups in $\underline{\underline{V}}$ form an infinite set. Hence there is a countably infinite set, say Δ, of finite groups in $\underline{\underline{V}}$ such that the orders of the chief-sections of the groups in Δ form an infinite set. Since $\underline{\underline{V}}$ is closed under the operation of taking homomorphic images, we may as well suppose that the orders of the minimal normal subgroups of the groups in Δ form an infinite set. From each $G \in \Delta$, select a minimal normal subgroup $N(G)$ of G, so that $\{N(G) : G \in \Delta\}$ is infinite. Let Ω be a (finite) set containing one copy of each (isomorphism type of) simple group in $\underline{\underline{V}}$. (Observe that $\underline{\underline{V}}$ is soluble if and only if each group in Ω is abelian.) Then each $G \in \Delta$ determines uniquely an element $B(G)$ in Ω, and a natural number $m(G)$, such that

$$N(G) \cong B(G)^{m(G)}.$$

Since Ω is a finite set, it contains an element, say B, such that

$$\{m(G) : B(G) = B,\ G \in \Delta\} \text{ is infinite.}$$

Put $\Delta_1 = \{G : B(G) = B, G \in \Delta\}$; since $\underline{\underline{V}}$ is jnC, it follows from Theorem 2.4.2 that

$$\underline{\underline{V}} = \mathrm{var}\Delta_1.$$

In case B is nonabelian, put $\Delta_2 = \{G/C_G(N(G)) : G \in \Delta_1\}$. Observe that every group in Δ_2 is monolithic with monolith isomorphic to a direct power of B. But because of the way we have chosen B, the orders of the monoliths of the groups in Δ_2 form an infinite set, and so we have a contradiction to Corollary 3.1.5.

Hence B is abelian, say of order p. Applying Lemma 2.4.1, we replace each group G in Δ_1 by G^*, and put $\Gamma = \{G^* : G \in \Delta_1\}$. Since $N(G^*) \sim N(G)$, the Lemma follows from Theorem 2.4.2 and Lemma 3.1.6 (i). //

3.2 The Theorem

The proof of the following theorem occupies the whole of this section.

3.2.1 THEOREM A jnC variety is reducible if and only if it is soluble of finite exponent.

The "if" part of Theorem 3.2.1 is easy to prove. For if $\underline{\underline{V}}$ is a jnC variety of finite exponent n which is also soluble of length ℓ, then $\underline{\underline{V}}$ is a subvariety of $(\underline{\underline{V}} \wedge \underline{\underline{A}}_n)^{\ell}$. Since $\underline{\underline{V}} \wedge \underline{\underline{A}}_n$ is a subvariety of $\underline{\underline{A}}_n$, which is Cross, it follows that $\underline{\underline{V}}$ is reducible.

Conversely, let $\underline{\underline{V}}$ be a reducible jnC variety. Then by Lemma 3.1.6 (i), $\underline{\underline{V}}$ is locally finite, say $\underline{\underline{V}}$ has (finite) exponent n. Moreover, by the same result, $\underline{\underline{V}}$ contains only finitely many (isomorphism classes of) finite simple groups; let Λ be a (finite) set containing one copy of each of them.

If $\underline{\underline{V}}$ is locally nilpotent, then by Lemma 3.1.6 (ii), $\underline{\underline{V}}$ is $\underline{\underline{A}}_p\underline{\underline{A}}_p$ for some prime p, and so $\underline{\underline{V}}$ is soluble of finite exponent. We shall suppose, therefore, that $\underline{\underline{V}}$ is not locally nilpotent, and consequently (Corollary 2.4.3) that there is a bound, say c, on the nilpotency class of nilpotent groups in $\underline{\underline{V}}$. Hence by Lemma 3.1.7 (and implicitly Corollary 3.1.5), there is a prime, say p, and a (countably) infinite set, say Γ, of pairwise-nonisomorphic monolithic groups in $\underline{\underline{V}}$ such that:

(i) $\underline{\underline{V}} = \mathrm{var}\Gamma$;

(ii) the monolith of each group in Γ is complemented, self-centralising, and has exponent p;

(iii) $\{|M(H)| : H \in \Gamma\}$ is an infinite set.

By [5, 1.2.2], a soluble group in $\underline{\underline{V}}$ has solubility length at most n_1c^2, where n_1 denotes the number of primes dividing n. Hence the soluble groups in $\underline{\underline{V}}$ form a subvariety, namely $\underline{\underline{V}} \wedge \underline{\underline{A}}^{nc^2}$. For a proof of Theorem 3.2.1 by contradiction, we assume that $\underline{\underline{V}} \wedge \underline{\underline{A}}^{nc^2}$ is a proper, and hence Cross, subvariety of $\underline{\underline{V}}$. Using Theorem 2.4.2, we may restate this assumption as follows:

(3.2.2) The orders of the chief-sections of the soluble groups in $\underline{\underline{V}}$ are bounded, say by d.

Now let $H \in \Gamma$, denote $M(H)$ by V, and let G be a complement for V in H. By [18, 52.24], and properties (ii) and (iii) of Γ,

$$\{|G| : H \in \Gamma\} \text{ is an infinite set.}$$

We may think of V as a faithful irreducible $\underline{F}(p)G$-module (G acts by conjugation), and so as a consequence of (3.2.2) we have:

(3.2.3) If A is a subgroup of G, U is an irreducible submodule of V_A, and $A/\ker U$ is soluble, then U has order at most d.

Before we can deduce the contradiction needed to establish Theorem 3.2.1, we need three lemmas.

Let S be the soluble radical of G, and suppose that

$$V_S = \bigoplus_{i=1}^{a(H)} U_i \ ; \quad U_i = \bigoplus_{j=1}^{b(H)} U_{ij}$$

is a Clifford decomposition of V_S. Let the kernel of U_{ij} be K_i.

3.2.4 LEMMA The sets $\{a(H) : H \in \Gamma\}$ and $\{|S| : H \in \Gamma\}$ are finite.

Proof. Suppose that $\{a(H) : H \in \Gamma\}$ is an infinite set. By Clifford's Theorem, $a(H)$ is the index of the inertia group of U_1 in G, and so the prime divisors of $a(H)$ all divide n. But n is finite, and so there is a prime, say r (which may be p), such that $\{a_r(H) : H \in \Gamma\}$ is infinite, where $a_r(H)$ is the r-share of $a(H)$. In particular, there is an element, say H_1, of Γ such that $a_r(H_1) > d$. Then if R_1 is a Sylow r-subgroup of G_1, Lemma 2.3.2 implies that the $\underset{\sim}{F}(p)$-dimension of an irreducible submodule of $V_{1_{S_1R_1}}$ is at least $a_r(H_1)$. Since S_1R_1 is a soluble subgroup of G_1, this contradicts (3.2.3).

Hence $\{a(H) : H \in \Gamma\}$ is finite, say $a(H) < a$ for all $H \in \Gamma$. Since S is irreducibly represented on U_{i1}, it follows from (3.2.3) that $|U_{i1}| \leq d$, and hence that $|S : K_i| \leq d!$, $1 \leq i \leq a(H)$. But V is faithful, and hence $\cap\{K_i : 1 \leq i \leq a(H)\}$ is trivial. It follows from Lemma 2.2.1 (ii) that $|S| < (d!)^a$. //

Suppose that $|S| < b$, for all $H \in \Gamma$, and denote $C_G(S)$ by C; then $\{|G : C| : H \in \Gamma\}$ is bounded by $b!$, and $\{|C| : H \in \Gamma\}$ is an infinite set. Let $M_1/C \cap S, \ldots, M_{m(H)}/C \cap S$ be the minimal normal subgroups of $G/C \cap S$ contained in $C/C \cap S$, and denote $M_1M_2 \ldots M_{m(H)}$ by M. Since $C \cap S$ is simultaneously the centre of S, the soluble radical of C, and the centre of C,

$M_i/C \cap S$ is nonabelian, $1 \leq i \leq m(H)$, and so there is a nonabelian simple group, say $B(i,H)$, in Λ such that $M_i/C \cap S$ is isomorphic to a direct power of $B(i,H)$.

3.2.5 LEMMA $\{|M_i| : 1 \leq i \leq m(H), H \varepsilon \Gamma\}$ is a finite set, whereas $\{|M| : H \varepsilon \Gamma\}$ is infinite. In particular, $\{m(H) : H \varepsilon \Gamma\}$ is an infinite set.

Proof. By Lemma 3.2.4, $\{|C \cap S| : H \varepsilon \Gamma\}$ is a finite set, whereas $\{|C| : H \varepsilon \Gamma\}$ is infinite; consequently $\{|C : C \cap S| : H \varepsilon \Gamma\}$ is infinite. Since $C \cap S$ is the soluble radical of C, and $M_i/C \cap S$ is isomorphic to a direct power of the nonabelian simple group $B(i,H)$, $C_{C/C\cap S}(M/C \cap S)$ avoids $M_i/C \cap S$, $1 \leq i \leq m(H)$, and hence is trivial. Thus $\{|M : C \cap S| : H \varepsilon \Gamma\}$ is an infinite set.

If $\{|M_i| : 1 \leq i \leq m(H), H \varepsilon \Gamma\}$ is an infinite set, then so is $\{|M_i : C \cap S| : 1 \leq i \leq m(H), H \varepsilon \Gamma\}$. But Λ is a finite set, and so it contains an element, say B, such that

$$\Omega = \{|M_i : C \cap S| : B(i,H) = B, 1 \leq i \leq m(H), H \varepsilon \Gamma\}$$

is an infinite set. (Observe that B is nonabelian.) Then if $C_{C/C\cap S}(M_i/C \cap S)$ is denoted by D_i,

$$\Delta = \{(C/C \cap S)/D_i : B(i,H) = B,\ 1 \leq i \leq m(H),\ H \in \Gamma\}$$

is a set of monolithic groups in $\underline{\underline{V}}$, and the monolith of each group in Δ is isomorphic to a direct power of B. Since Ω is an infinite set, we have a contradiction to Corollary 3.1.5. Hence $\{|M_i| : 1 \leq i \leq m(H),\ H \in \Gamma\}$ is a finite set. //

By Lemma 3.2.5, we can choose H in Γ so that $2^{m(H)} > d$.

3.2.6 LEMMA Each of $M^{(1)}$ and $M_i^{(1)}$, $1 \leq i \leq m(H)$ are perfect. Furthermore, $M^{(1)}$ is a central product of $M_1^{(1)},\ldots,M_{m(H)}^{(1)}$.

Proof. Since $S \cap C$ is central in M_i, and $M_i/S \cap C$ is isomorphic to a direct power of the nonabelian simple group $B(i,H)$, it follows from Lemma 2.2.1 (iii) that $M_i^{(1)}$ is perfect, $1 \leq i \leq m(H)$. Similarly, $M^{(1)}$ is perfect, since $M/C \cap S$ is the direct product of $M_1/C \cap S,\ldots,M_{m(H)}/C \cap S$.

For the second part, we have to show that if $i \neq j$, then $M_j^{(1)} \leq C_{M^{(1)}}(M_i^{(1)})$, and also that $M^{(1)} = \langle M_1^{(1)},\ldots,M_{m(H)}^{(1)}\rangle$. Let $g \in M_j^{(1)}$, and define a map $\alpha(g) : M_i^{(1)} \to S \cap C$ by

$$h\alpha(g) = [h,g].$$

Now $\alpha(g)$ is a homomorphism, for

$$\begin{aligned}(h_1h_2)\alpha(g) &= [h_1h_2,g],\\ &= [h_1,g][h_2,g], \qquad \text{since} \quad C \cap S = Z(C),\\ &= h_1\alpha(g)h_2\alpha(g).\end{aligned}$$

But $M_i^{(1)}$ is perfect, and $C \cap S$ is abelian, and so $M_i^{(1)}$ is the kernel of $\alpha(g)$; that is, g centralises $M_i^{(1)}$. Since $M = M_1M_2\ldots M_{m(H)}$, and $M_i = M_i^{(1)}S \cap C$, it follows that $M = M_1^{(1)}M_2^{(1)}\ldots M_{m(H)}^{(1)}\ S \cap C$, and hence that $M^{(1)} = M_1^{(1)}\ldots M_{m(H)}^{(1)}$. //

Now $M^{(1)}$ is a normal subgroup of G and V is a faithful irreducible $\underline{F}(p)G$-module, and so by Clifford's Theorem, if L is the kernel of an irreducible submodule X of $V_{M^{(1)}}$,

$$\cap\{L^g : g \in G\} = E.$$

Since $M_i^{(1)}$ is also normal in G,

$$L \not\geq M_i^{(1)}, \quad 1 \leq i \leq m(H).$$

Let $\underline{E}$ be the field obtained from $\underline{F}(p)$ by adjoining to it all the primitive n^{th} roots of unity. Since the exponent of G divides n, it follows from Theorem 2.3.5 that $\underline{E}$ is a splitting field for G. Moreover, $\underline{E}$ is a finite normal extension of (the perfect field) $\underline{F}(p)$, and so by Theorem 2.3.6, $X^{\underline{E}}$ is completely reducible, and the irreducible components of $X^{\underline{E}}$ are all Galois conjugate. Thus if U is an irreducible component of $X^{\underline{E}}$, the kernel of U is L. By Corollary 2.3.10 (and Lemma 2.3.3 if $m(H) > 2$),

$$U \cong U_1 \# \ldots \# U_{m(H)},$$

where U_i is a (necessarily absolutely) irreducible submodule of $U_{M_i^{(1)}}$. Since $L \not\geq M_i^{(1)}$, the kernel L_i of U_i is a proper normal subgroup of $M_i^{(1)}$, and so $M_i^{(1)}/L_i$ is non-trivial perfect. In particular, $M_i^{(1)}/L_i$ is not a p-group, $1 \leq i \leq m(H)$. It follows from Theorem 2.3.1 that there is a subgroup, say A_i, of $M_i^{(1)}$ containing L_i, and an irreducible submodule, say W_i, of ${U_i}_{A_i}$, such that A_i/L_i is soluble and W_i has $\underline{E}$-dimension at least two. Since the kernel N_i of W_i contains L_i, A_i/N_i is soluble also. If $A = \langle A_i : 1 \leq i \leq m(H)\rangle$, then A is the central produce of $A_1, \ldots, A_{m(H)}$, and so it follows from Corollary 2.3.10 that $W_1 \# \ldots \# W_{m(H)}$ is isomorphic to an irreducible submodule, say W, of U_A. Observe that the $\underline{E}$-dimension of W is at least $2^{m(H)}$, and that the kernel N of W contains $\langle N_i : 1 \leq i \leq m(H)\rangle$. But $A/\langle N_i : 1 \leq i \leq m(H)\rangle$ is a homomorphic image of

$\Pi\{A_i/N_i : 1 \leq i \leq m(H)\}$, and so A/N is soluble. Now W is an irreducible submodule of $V\underline{E}_A$ which is the same thing as $V_A\underline{E}$, and so by the Jordan-Hölder Theorem, there is a composition factor, say W_1, of V_A such that W is isomorphic to a composition factor of $W_1\underline{E}$. By Theorem 2.3.6 though, $W_1\underline{E}$ is completely reducible, and its irreducible components are all Galois conjugate; hence N is also the kernel of W_1. But

$$|W_1| \geq |W| \geq 2^{m(H)} > d,$$

and A/N is soluble. This contradicts (3.2.3), and so the proof of Theorem 3.2.1 is complete. //

CHAPTER 4

A CHARACTERISATION OF $\mathfrak{A}_p\mathfrak{T}_q$

Our main purpose in this Chapter is to prove (as Theorem 4.3.1) that if m and n are coprime positive integers, the jnC subvarieties of $\mathfrak{A}_m(\mathfrak{N}_2 \wedge \mathfrak{B}_n)$ are precisely the elements of $\{\mathfrak{A}_p\mathfrak{T}_q : p \text{ and } q \text{ prime, } p \text{ divides } m, q \text{ divides } n\}$. This occupies section 4.3, and uses the results obtained in sections 4.1 and 4.2.

The ideas described in this Chapter date from the first year of my project; my first proof of Theorem 4.3.1 being completed by December 1968. Since that time, wholesale changes have been made to the proofs of various steps in the argument, and the work has been published as my contribution to a joint paper [2] by myself, R.A. Bryce and John Cossey (their contribution was to prove that the subvarieties of $\mathfrak{A}_m(\mathfrak{N}_2 \wedge \mathfrak{B}_n)$ are finitely-based whenever m and n are coprime). I shall try to indicate (wherever this is possible) what remains of my original work. Broadly speaking, sections 4.1 and 4.3 are substantially as I found them, whereas section 4.2 is not; that is, the proof of Theorem 4.2.1 is not. In particular, I acknowledge the ideas of R.A. Bryce and John Cossey, which led to Theorem 2.4 of [2] clearing up a particularly messy part of my work. Finally, the proof of Theorem 4.2.1 given here improves that given in the joint paper by incorporating some ideas of L.G. Kovács.

4.1 Groups in $\underline{\underline{N}}_2$ with cyclic centre

As we have mentioned earlier, the proof of Theorem 4.3.1 relies on an application of the results of Graham Higman which were sketched in section 2.4, in the case $\underline{\underline{B}} = (\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_n)$. Thus Theorems 2.4.4 and 2.4.6 lead us to consider irreducible linear groups in $\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_n$ over $\underline{F}(p)$, where p does not divide n.

If K is an irreducible linear group in $\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_n$ acting on a space V over $\underline{F}(p)$, and C is the centraliser of K in $\mathrm{End}_{\underline{F}(p)}(V)$, then by Schur's Lemma [4,27.3], C is a (finite) skewfield; that is, C is a field. But the centre $Z(K)$ of K is a subgroup of (the multiplicative group of) C, and hence is cyclic. Thus the aim of this section is to classify, up to isomorphism, the groups in $\underline{\underline{N}}_2$ with cyclic centre. Since a nilpotent group is the direct product of its Sylow subgroups, we confine our attention to q-groups. (Observe that since each q-group with cyclic centre is monolithic, it has faithful irreducible representations over $\underline{F}(p)$: hence our exercise is no more general than it needs to be.)

4.1.1 THEOREM Let K be a q-group in $\underline{\underline{N}}_2$ with $K^{(1)}$ cyclic. Then K is a central product of a central subgroup and a subgroup which is itself a central product of nonabelian two-generator subgroups.

Proof. Let $d(K)$ be n. If either K is abelian or $n \leq 2$, there is nothing to prove; so we shall assume that K is nonabelian and $n > 2$. Let $\{x_1,\ldots,x_n\}$ be a minimal generating set for K; then since $K \varepsilon \underline{\underline{N}}_2$,

$$K^{(1)} = \langle [x_i,x_j] : 1 \leq i < j \leq n \rangle.$$

But $K^{(1)}$ is nontrivial and cyclic, and so we may suppose without loss of generality that

$$K^{(1)} = \langle [x_1,x_2] \rangle.$$

Thus for all i, $2 < i \leq n$, there are elements $s(i)$, $t(i)$ of $\underline{N}$ such that

$$[x_1,x_i] = [x_1,x_2]^{s(i)}, \quad [x_2,x_i] = [x_1,x_2]^{t(i)}.$$

Put y_i equal to $x_i x_1^{t(i)} x_2^{-s(i)}$, $2 < i \leq n$, and denote $\langle y_3,\ldots,y_n \rangle$ by L; then K is a central product of L and $\langle x_1,x_2 \rangle$. Observe that $L^{(1)}$ is cyclic, and is amalgamated with as large a subgroup of $\langle x_1,x_2 \rangle^{(1)}$ as possible. Since $d(L) = n - 2$, the proof can be completed by induction. Notice that in case n is odd, the central subgroup is always nontrivial. //

Before we state a Corollary to Theorem 4.1.1, we remark that a q-group in $\underline{\underline{N}}_2$ is critical if and only if it is either cyclic or a nonabelian two-generator group with cyclic centre [18, 51.35, 51.36, and the sentence following 51.36].

4.1.2 COROLLARY A q-group in $\underline{\underline{N}}_2$ with cyclic centre is a central product of nonabelian critical subgroups, possibly together with a (central) cyclic (that is, abelian critical) subgroup. Whenever such a group is written in this way, the amalgamations of the critical subgroups must be as large as possible.

Proof. Suppose (in the proof of Theorem 4.1.1) that $Z(K)$ is cyclic. Since L centralises $\langle x_1,x_2\rangle$, it follows that both L and $\langle x_1,x_2\rangle$ have cyclic centres. Thus $\langle x_1,x_2\rangle$ is critical and nonabelian, and the amalgamation between $Z(L)$ and $Z(\langle x_1,x_2\rangle)$ must be as large as possible. //

When a group in $\underline{\underline{N}}_2$ with cyclic centre is written as a central product of critical groups as prescribed by Corollary 4.1.2, we say that it is centrally decomposed. We call the critical groups occuring in such a central decomposition the central components. The rest of this section is in two parts. In Theorem 4.1.4 (which uses Lemma 4.1.3), we classify all nonabelian critical groups in $\underline{\underline{N}}_2$. From then on we shall be concerned with the question of whether or not a given list of critical groups can give rise to nonisomorphic groups with cyclic centres because of different amalgamations.

4.1.3 LEMMA Let G be a group in $\underline{\underline{N}}_2$, g, h elements of G. Then for all $\ell \geq 2$, $(gh)^\ell = g^\ell h^\ell [h,g]^{\binom{\ell}{2}}$.

Proof. The proof is by induction on ℓ. Observe first that

$$\begin{aligned}(gh)^2 &= ghgh = g^2h[h,g]h,\\ &= g^2h^2[h,g],\end{aligned}$$

since $G \varepsilon \underline{\underline{N}}_2$. If $(gh)^\ell = g^\ell h^\ell [h,g]^{\binom{\ell}{2}}$,

$$\begin{aligned}(gh)^{\ell+1} &= (gh)^\ell(gh),\\ &= g^\ell h^\ell [h,g]^{\binom{\ell}{2}} gh,\\ &= g^\ell h^\ell gh[h,g]^{\binom{\ell}{2}}, \text{ since } G \varepsilon \underline{\underline{N}}_2\\ &= g^{\ell+1}h^{\ell+1}[h,g]^{\ell+\binom{\ell}{2}}.\end{aligned}$$

Since $\ell + \binom{\ell}{2} = \binom{\ell+1}{2}$, the proof is complete. //

We now define for each prime q and each $k,\ell \in \underline{P}$ with $k \geq \ell$ a group $Q(q^k,q^\ell)$, as follows:

if $k \geq 2\ell$, $Q(q^k,q^\ell) = \langle g,h : h^{q^\ell} = e, g^{q^{k-\ell}} = [g,h]\rangle$.

If $k \leq 2\ell$, $Q(q^k,q^\ell) = \langle g,h : h^{q^\ell} = [g,2h] = [h,2g] = e$,

$g^{q^\ell} = [g,h]^{q^{2\ell-k}}\rangle$.

Again, for each $\ell \in \underline{P}$ we define $R(2^\ell)$ by:

$$R(2^\ell) = \langle g,h : g^{2^\ell} = h^{2^\ell} = [g,h]^{2^{\ell-1}}, [g,2h] = [h,2g] = e\rangle.$$

4.1.4 THEOREM If G is a nonabelian critical group in $\underline{\underline{N}}_2$ with exponent q^k and derived exponent q^ℓ, then G is isomorphic to $Q(q^k,q^\ell)$, except that if $q = 2$ and $k = \ell + 1$ and every generator of G has order $2^{\ell+1}$, then G is isomorphic to $R(2^\ell)$.

4.1.5 REMARK We remark that it follows from Lemma 4.1.3 that if G is a critical 2-group, the derived exponent of G is less than the exponent of G, and so we need not have defined $Q(2^\ell,2^\ell)$. We omit the elementary but somewhat tedious proof of the fact that the

$Q(q^k,q^\ell)$ and the $R(2^\ell)$ are all pairwise-nonisomorphic critical groups except that $Q(2^\ell,2^\ell) \cong Q(2^{\ell+1},2^\ell)$. All we shall need in this direction in the proof of Theorem 4.3.1 are the simple facts that $Q(q,q)$ is nonabelian of order q^3, and that $Q(q,q)$ has exponent q for odd q, while $Q(2,2)$ is the dihedral group $Q(4,2)$ of order eight.

Proof of Theorem 4.1.4. Let G be a nonabelian two-generator q-group in $\underline{\underline{N}}_2$ with $Z(G)$ cyclic, say G has exponent q^k. If there is an element x in $G - D(G)$ such that

$x^{q^{k-1}} = e$, then $|G^{(1)}| \leq q^{k-1}$, and so by Lemma 4.1.3, for all

$y \in G$, $c \in G^{(1)}$, $(y^q c)^{q^{k-1}} = e$; that is, $D(G)$ has exponent dividing q^{k-1} [9,iii, 3.14]. Thus we can always find an element, say g, in $G - D(G)$ such that $|g| = q^k$. Let $\{g,h\}$ be a generating set for G such that $|g| = q^k$, and suppose that $|h| = q^m$ and $|[g,h]| = q^\ell$. Since

$$[g,h]^{q^m} = [g,h^{q^m}] = e,$$

$\ell \leq m \leq k$. Now $g^{q^n} \in Z(G)$ if and only if $[g^{q^n},h] = e$; that is, if and only if $n \geq \ell$. Similarly, $h^{q^n} \in Z(G)$ if and only if $n \geq \ell$. But every element of G can be written in the form $h^r g^s [g,h]^t$, and

$$h^r g^s [g,h]^t \in Z(G) \iff h^r g^s \in Z(G), \quad \text{since } G \in \underline{\underline{N}}_2,$$

$$\iff h^r, g^s \in Z(G), \quad \text{since } G = \langle g,h \rangle.$$

Thus

$$Z(G) = \langle g^{q^\ell}, h^{q^\ell}, [g,h] \rangle.$$

But $Z(G)$ is cyclic, and so since $|g| \geq |h|$, two possibilities arise, namely $Z(G) = \langle g^{q^\ell} \rangle$ if $k \geq 2\ell$, and $Z(G) = \langle [g,h] \rangle = G^{(1)}$ if $k \leq 2\ell$.

<u>CASE 1</u>. $Z(G) = \langle g^{q^\ell} \rangle$. There is an integer s such that

$$[g,h] = g^{sq^\ell}.$$

Hence $2\ell \leq k$, and for all $n \geq 0$,

$$[g,h]^{q^n} = g^{sq^{\ell+n}}.$$

Consequently $g^{sq^{2\ell}} = e$ while $g^{sq^{2\ell-1}} \neq e$. It follows that $s = tq^{k-2\ell}$, where q does not divide t. Choose t' such that $tt' \equiv 1(\text{mod} q^\ell)$; then

$$[g,h^{t'}] = [g,h]^{t'} = g^{q^{k-\ell}}.$$

Since q does not divide t', $G = \langle g,h^{t'} \rangle$, and so we may as well suppose that

$$[g,h] = g^{q^{k-\ell}}. \qquad (*)$$

Recall that $h^{q^\ell} \varepsilon Z(G)$, so that there is an integer u with

$$h^{q^\ell} = g^{uq^\ell}.$$

Observe that if h is replaced by $h^* = hg^v$, then $(*)$ becomes $[g,h^*] = g^{q^{k-\ell}}$. We shall try to choose v so that $|hg^v| = q^\ell$.

By Lemma 4.1.3, $(hg^v)^{q^n} \varepsilon Z(G)$ if and only if $n \geq \ell$; in particular, $|hg^v| \geq q^\ell$, for all v. Now by Lemma 4.1.3,

$$(hg^v)^{q^\ell} = h^{q^\ell} g^{vq^\ell} [g,h]^{v(q^\ell-1)q^\ell/2},$$

$$= g^{uq^\ell + v(q^\ell + q^k(q^\ell-1)/2)},$$

$$= g^{q^\ell(u+v(1+q^{k-\ell}(q^\ell-1)/2)}.$$

Thus a v is of the desired kind if and only if it satisfies the congruence

$$u + v(1+q^{k-\ell}(q^\ell-1)/2) \equiv 0(\text{mod} q^{k-\ell}). \qquad (**)$$

If q is odd, we can simply choose $v = -u$. If $q = 2$, $(**)$

reduces to

$$u + v(1-2^{k-\ell-1}) \equiv 0 (\mathrm{mod}\, 2^{k-\ell}).$$

This can always be solved for v except when $k = \ell + 1$ and u is not zero mod $2^{k-\ell}$; since $k \geq 2\ell$, this means that $\ell = 1$ and $k = 2$. Whenever a v can be found such that $|hg^v| = q^\ell$, G is a homomorphic image of $Q(q^k,q^\ell)$. But $Q(q^k,q^\ell)$ has cyclic centre, and $|Q(q^k,q^\ell)^{(1)}| \leq q^\ell$. It follows that if such a G exists, $|Q(q^k,q^\ell)^{(1)}| = q^\ell$, and the kernel of the homomorphism from $Q(q^k,q^\ell)$ to G is trivial; that is, $G \cong Q(q^k,q^\ell)$. If no such v can be found, $|hg^v| = 4$, for all v. Thus $h^4 = g^4 = e$ and $h^2 = g^2 = [g,h]$. It follows that G is a nonabelian homomorphic image of $R(2)$ (the quaternion group of order eight), and hence that $G \cong R(2)$.

CASE 2. $Z(G) = \langle [g,h] \rangle = G^{(1)}$. The argument proceeds along lines similar to those in case 1. There is an integer s such that

$$[g,h]^s = g^{q^\ell}.$$

Hence $2\ell \geq k$, and for all $n \geq 0$,

$$[g,h]^{sq^n} = g^{q^{\ell+n}}.$$

Consequently $[g,h]^{sq^{k-\ell-1}} \neq e$, while $[g,h]^{sq^{k-\ell}} = e$. It follows that $s = tq^{2\ell-k}$, where q does not divide t. Since $G = \langle g,h^t \rangle$, and $[g,h^t]^{q^{2\ell-k}} = g^{q^\ell}$, we may as well suppose that

$$[g,h]^{q^{2\ell-k}} = g^{q^\ell} \qquad (@)$$

Recall that $h^{q^\ell} \in Z(G)$, so that there is an integer u with

$$h^{q^\ell} = [g,h]^u.$$

Since $|h| = q^m$, $u = aq^{2\ell-m}$, where q does not divide a. We again observe that if h is replace by $h^* = hg^v$, for any integer v, (@) becomes $[g,h^*]^{q^{2\ell-k}} = g^{q^\ell}$. We also note that $|hg^v| \geq q^\ell$, for all v. We shall try to choose v such that $|hg^v| = q^\ell$. By Lemma 4.1.3,

$$(hg^v)^{q^\ell} = h^{q^\ell} g^{vq^\ell} [g,h]^{vq^\ell(q^\ell-1)/2},$$

$$= [g,h]^{u+v(q^{2\ell-k}+q^\ell(q^\ell-1)/2)}.$$

Thus we can find a suitable v if and only if we can solve the congruence

$$u + v(q^{2\ell-k} + q^{\ell}(q^{\ell}-1)/2) \equiv 0(\mathrm{mod}\, q^{\ell}).$$

If q is odd, we have to solve (remembering $u = aq^{2\ell-m}$, where q does not divide a)

$$aq^{2\ell-m} + vq^{2\ell-k} \equiv 0(\mathrm{mod}\, q^{\ell}),$$

which we can do by choosing $v = -aq^{k-m}$. If $q = 2$, we have to solve

$$a2^{2\ell-m} + v(2^{2\ell-k} - 2^{\ell-1}) \equiv 0(\mathrm{mod}\, 2^{\ell}).$$

This can be done by choosing $v = 0$ if $\ell = m$ and $v = -a2^{k-m}/(1-2^{k-\ell-1})$ if $\ell < m$, unless, in the latter case $k = \ell + 1$. Whenever a suitable v can be chosen, G is a homomorphic image of $Q(q^k,q^{\ell})$, with $2\ell \geq k$. Thus if such a G exists, $|Q(q^k,q^{\ell})^{(1)}| = q^{\ell}$, and since $Z(Q(q^k,q^{\ell}))$ is cyclic, the homomorphism is an isomorphism; that is $G \cong Q(q^k,q^{\ell})$. If a v cannot be found, then since $q = 2$ and $k = \ell + 1$, for all v, $|hg^v| = 2^{\ell+1}$. Thus $m = \ell + 1$, and we have $g^{2^{\ell}} = h^{2^{\ell}} = [g,h]^{2^{\ell-1}}$. Hence G is a homomorphic image of $R(2^{\ell})$. Again $|R(2^{\ell})^{(1)}| \leq 2^{\ell}$, so that if such a G exists $|R(2^{\ell})^{(1)}| = 2^{\ell}$, and $G \cong R(2^{\ell})$. //

A presentation of a nonabelian critical group G in $\underset{=}{N}_2$ is called a canonical presentation if the defining relations are those of a $Q(q^k,q^\ell)$ or $R(2^\ell)$ to which G is isomorphic.

4.1.6 COROLLARY Let G be a group in $\underset{=}{N}_2$ with cyclic centre. Then every automorphism of $Z(G)$ is the restriction to $Z(G)$ of an automorphism of G.

Proof. (i) Since a nilpotent group is the direct product of its Sylow subgroups, each of which is a characteristic subgroup, the Corollary is true if and only if it is true for q-groups in $\underset{=}{N}_2$ with cyclic centre.

(ii) Suppose G is a critical q-group in $\underset{=}{N}_2$; so that either G is isomorphic to one of the groups listed immediately before the statement of Theorem 4.1.4, or G is abelian. The Corollary is trivial if G is abelian, so suppose that this is not the case. If

$Z(G) = \langle z : z^{q^n} = e\rangle$, then $\mathrm{Aut}Z(G) = \{\alpha_t : 0 < t < q^n,\ q$ does not divide $t\}$ where $z\alpha_t = z^t$. If g,h are generators for G in a canonic presentation of G, check that $g\beta_t = g^t$, $h\beta_t = h$ defines an automorphism β_t of G; then the restriction of β_t to $Z(G)$ is α_t.

(iii) Finally, suppose that G is a q-group in $\underline{\underline{N}}_2$ with cyclic centre. By Corollary 4.1.2, G can be written as a central product of critical subgroups, say $G_1,\ldots,G_n$, of which at most one is abelian. Moreover, we can number the G_i so that

$$Z(G) = Z(G_1) \geq Z(G_2) \geq \ldots \geq Z(G_n).$$

Suppose $\alpha \in \mathrm{Aut}Z(G)$, and denote the restriction of α to $Z(G_i)$ by α_i. By (ii), there is an automorphism, say β_i, of G_i whose restriction to $Z(G_i)$ is α_i. Define the automorphism β of G by requiring the restriction of β to G_i to be β_i. //

4.1.7 LEMMA Let K and L be groups, and suppose that G and H are central products of K and L amalgamating $Z(K)$ and the subgroup Y of $Z(L)$. If every automorphism of $Z(K)$ is the restriction to $Z(K)$ of an automorphism of K, then G is isomorphic to H.

Proof. There are isomorphisms $\mu,\nu : Z(K) \to Y$, such that if $M = \{(z,z\mu) : z \in Z(K)\}$ and $N = \{(z,z\nu) : z \in Z(K)\}$, $G \cong K{\times}L/M$ and $H \cong K{\times}L/N$. Now $\mu\nu^{-1} \in \mathrm{Aut}Z(K)$, and so there is an automorphism, say ρ, of K whose restriction to $Z(K)$ is $\mu\nu^{-1}$. Define $\sigma \in \mathrm{Aut}(K{\times}L)$ by $(k,\ell)\sigma = (k\ell,\ell)$; then for all $z \in Z(K)$

$$(z,z\mu)\sigma = (z\rho,z\mu) = (z\mu\nu^{-1},z\mu),$$
$$= (z\mu\nu^{-1},(z\mu\nu^{-1})\nu).$$

Hence $M\sigma = N$, and so $G \cong H$. //

4.1.8 COROLLARY Let G and H be groups in $\underline{\underline{N}}_2$ with cyclic centres. Then G is isomorphic to H if and only if they have central decompositions which have the same list of central components.

Proof. Observe that the Corollary is true if and only if it is true for q-groups. The "only if" part is trivial, and if G and H are critical, so is the "if" part. Thus we may suppose that G and H are not critical, are q-groups, and that each has a central decomposition in which the list of central components is $K_1,\dots,K_n$. We can suppose without loss of generality that the K_i have been numbered so that $|Z(K_1)| \leq \dots \leq |Z(K_n)|$. By Corollary 4.1.6 and Lemma 4.1.7, the subgroups G_2 of G and H_2 of H generated by K_1 and K_2 are isomorphic (since there is a unique subgroup of $Z(K_2)$ whose order is $|Z(K_1)|$). Again by Corollary 4.1.6 and Lemma 4.1.7, the subgroups of G and H generated by G_2 and K_3 and H_2 and K_3 are isomorphic. Continuing in this way, we see eventually that G is isomorphic to H. //

4.1.9 REMARK Whereas it follows from Corollary 4.1.8 that the list of central components occuring in a central decomposition of a group G in $\underline{\underline{N}}_2$ with cyclic centre determine G (up to isomorphism), it is not generally the case that such a group G has only one central decomposition. For example, let g_1, h_1 and g_2, h_2 be the generators in canonical presentations for $R(2)$ (the quaternion group of order eight), and let G be the central product of $\langle g_1, h_1 \rangle$ and $\langle g_2, h_2 \rangle$ (amalgamating, of course, $[g_1, h_1]$ and $[g_2, h_2]$). Then $\langle g_1, h_1 g_2 \rangle \cong \langle g_2, g_1 h_2 \rangle \cong Q(4,2)$ (the dihedral group of order eight), and G is the central product of $\langle g_1, h_1 g_2 \rangle$ and $\langle g_2, h_1 g_2 \rangle$.

4.2 Representations of groups in $\underline{\underline{N}}_2$ with cyclic centre

In this section we prove the following Theorem ([2, 2.4]):

4.2.1 THEOREM Let K be a group in $\underline{\underline{N}}_2$ with cyclic centre, and let $\underline{E}$ be a perfect field whose characteristic does not divide $|K|$. Then the faithful irreducible representations of K over $\underline{E}$ are all linearly equivalent; that is, there is only one linear isomorphism class of linear groups which consists of groups (abstractly) isomorphic to K.

We need a lemma ([2, 2.9]):

4.2.2 LEMMA Let K be a group in $\underline{\underline{N}}_2$ with cyclic centre of order q^{ℓ}, and let $\underline{E}$ be an algebraically closed field whose characteristic is not q. Then there are $q^{\ell-1}(q-1)$ inequivalent faithful irreducible representations of K over $\underline{E}$.

Proof. Observe that K must be a q-group. Let m_1 be the number of distinct conjugacy classes of K; then m_1 is the number of inequivalent irreducible representations of K over $\underline{E}$ [4, 27.22]. Now K is monolithic, and so a representation of K over $\underline{E}$ is not faithful if and only if its kernel contains $M(K)$. Hence if m_2 is the number of distinct conjugacy classes of $K/M(K)$, the number of inequivalent faithful irreducible representations of K over $\underline{E}$ is $m_1 - m_2$. In case K is abelian, it is cyclic of order q^{ℓ}, so that $m_1 = q^{\ell}$, $m_2 = q^{\ell-1}$, and the Lemma is proved. Suppose K is nonabelian, and let x be a non-central element of K conjugate modulo $M(K)$ to y. Then there is $k \in K$, $z \in M(K)$, such that

$$y^k = xz.$$

But $M(K) \leq M(K^{(1)})$, so there is $k_1 \in K$ such that $[k_1,x] = z$.

Hence

$$x = y^{kk_1}.$$

It follows that if n is the number of non-central conjugacy classes of K, then $m_1 = n + |Z(K)|$ and $m_2 = n + |Z(K) : M(K)|$. Thus $m_1 - m_2$ is $q^{\ell} - q^{\ell-1}$, as required. //

We can now prove Theorem 4.2.1 in two steps:

(i) Suppose that K is a q-group in $\underline{\underline{N}}_2$ with cyclic centre, and $\underline{E}$ is an algebraically closed field of characteristic not q. If $|Z(K)| = q^{\ell}$, then $|\mathrm{Aut}Z(K)| = q^{\ell} - q^{\ell-1}$ (since if G is cyclic, $|\mathrm{Aut}G| = |G| - |D(G)|$). Denote $q^{\ell} - q^{\ell-1}$ by s, and suppose that $\mathrm{Aut}Z(K) = \{\beta_1, \ldots, \beta_s\}$. By Corollary 4.1.6, there is an automorphism, say α_i, of K such that the restriction of α_i to $Z(K)$ is β_i, $1 \leq i \leq s$. Since K is monolithic and the regular representation of K over $\underline{E}$ is completely reducible (by Maschke's Theorem), K has a faithful irreducible representation, say $\tau : K \to GL(n, \underline{E})$, over $\underline{E}$. We shall show that the faithful irreducible representations $\alpha_i\tau$, $1 \leq i \leq s$, of K over $\underline{E}$ are linearly equivalent but pairwise-inequivalent. Firstly, they are linearly equivalent, for

$$K(\alpha_i\tau) = (K\alpha_i)\tau = K\tau \leq GL(n, \underline{E}), \; 1 \leq i \leq s.$$

Suppose $Z(K) = \langle z : z^{q^\ell} = e\rangle$; since $\underline{E}$ is algebraically closed and τ is irreducible, it follows from Schur's Lemma [4, 27.22] that $z\alpha_i\tau$ is a scalar matrix and hence is conjugate to no other matrix in $GL(n,\underline{E})$. But since τ is faithful, $z\alpha_i\tau = z\alpha_j\tau$ if and only if $z\alpha_i = z\alpha_j$, that is if and only if $i = j$, and so $\{\alpha_i\tau : 1 \leq i \leq s\}$ is a set of s pairwise-inequivalent faithful irreducible representations of K over $\underline{E}$. It now follows from Lemma 4.2.2 that the faithful irreducible representations of K over $\underline{E}$ are all linearly equivalent.

(ii) K is a group in $\underline{\underline{N}}_2$ with cyclic centre, and $\underline{E}$ is a perfect field whose characteristic does not divide $|K|$.

Denote the algebraic closure of $\underline{E}$ by $\underline{E}^*$, and let $\sigma : K \to GL(n,\underline{E})$ be a faithful irreducible representation of K over $\underline{E}$. Then by Maschke's Theorem, $\sigma^{\underline{E}^*}$ is completely reducible. Since K is monolithic, and

$$\ker\sigma^{\underline{E}^*} = \ker\sigma = E,$$

$\sigma^{\underline{E}^*}$ has a faithful irreducible component, say τ. Thus by Theorem 2.3.8, it suffices to prove Theorem 4.2.1 in case $\underline{E}$ is algebraically closed. Since K is nilpotent, it is the direct product of its Sylow subgroups, each of which has cyclic centre. Then by Corollary 2.3.11, it suffices to prove Theorem 4.2.1 in case K is a q-group. As this is the content of (i), the proof of Theorem 4.2.1 is complete. //

4.3 The jnC subvarieties of $\mathbf{A}_m(\mathbf{N}_2 \wedge \mathbf{B}_n)$, m and n coprime

The proof of the following theorem occupies the whole of this section.

4.3.1 THEOREM If m and n are coprime positive integers, the jnC subvarieties of $\mathbf{A}_m(\mathbf{N}_2 \wedge \mathbf{B}_n)$ are precisely the elements of $\{\mathbf{A}_p\mathbf{T}_q : p \text{ and } q \text{ prime, } p \text{ divides } m, \ q \text{ divides } n\}$.

Proof. By Theorem 2.4.5, $\mathbf{A}_p\mathbf{T}_q$ is a jnC variety whenever p divides m and q divides n. It is also a subvariety of $\mathbf{A}_m(\mathbf{N}_2 \wedge \mathbf{B}_n)$ [18, 21.21 and 21.24]. On the other hand, let $\mathbf{V}$ be a jnC subvariety of $\mathbf{A}_m(\mathbf{N}_2 \wedge \mathbf{B}_n)$. By Theorem 2.4.6, there is a prime divisor, say p, of m such that $\mathbf{V}$ is a subvariety of $\mathbf{A}_p(\mathbf{N}_2 \wedge \mathbf{B}_n)$. Now

$$(\mathbf{N}_2 \wedge \mathbf{B}_n \wedge \mathbf{V}) < \mathbf{V} \leq \mathbf{A}_p(\mathbf{N}_2 \wedge \mathbf{B}_n \wedge \mathbf{V}),$$

and so by Theorem 2.4.4, there is a closed class, say $\mathbf{X}$, of irreducible linear groups in $\mathbf{N}_2 \wedge \mathbf{B}_n$ over $F(p)$ such that $L(\mathbf{V}) = \mathbf{X}$. Since $\mathbf{V}$ is a jnC variety, $\mathbf{X}$ contains infinitely many (isomorphism classes of) linear groups, whereas every proper closed subclass of $\mathbf{X}$ contains only finitely many. (By Theorem 4.2.1, we don't need to specify whether by the word "isomorphism" we mean "linear isomorphism" or "abstract isomorphism".) Each group K in $\underline{\underline{X}}$ has cyclic centre (see the opening paragraph of section 4.1), and belongs to $\underline{\underline{N}}_2$; hence by Corollary 4.1.2, K can be written as a central product of critical subgroups. By Theorem 4.1.4, there are only finitely many (isomorphism classes of) critical groups in $\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_n$; let $\{L_1,\ldots,L_t\}$ contain precisely one copy of each. Suppose for each group $K \in \underline{\underline{X}}$ we are given a central decomposition of K; then define t functions $\alpha_1,\ldots,\alpha_t : \underline{\underline{X}} \to \underline{N}$ by: $\alpha_i(K)$ is the number of times a central component of the given central decomposition of K is isomorphic to L_i. By Corollary 4.1.2, if L_i is abelian, $\alpha_i(\underline{\underline{X}})$ is a subset of $\{0,1\}$. Since K is determined up to isomorphism by $\alpha_1(K),\ldots,\alpha_t(K)$ (Corollary 4.1.8), and $\underline{\underline{X}}$ contains infinitely many isomorphism classes, for some i, $\alpha_i(\underline{\underline{X}})$ is an infinite subset of $\underline{N}$. Certainly L_i is nonabelian, and we may suppose without loss of generality that $i = 1$; denote L_1 by Q, and the (unique) prime divisor of $|Q|$ by q.

Let $K \in \underline{\underline{X}}$, and let $s \in \underline{N}$, $s \leq \alpha_1(K)$. Suppose L is a subgroup of K generated by any s of the $\alpha_1(K)$ central components of K (in the given central decomposition) which are isomorphic to Q. By Corollary 4.1.8, the isomorphism type of L is unaffected by the particular choice of components. Suppose that K acts on the $\underline{F}(p)$-space V; then V_L is homogeneous (Lemma 2.3.3). Thus an irreducible component V_0 of V_L is faithful, and so since $\underline{\underline{X}}$ is closed, the restriction of L to V_0 belongs to $\underline{\underline{X}}$. It follows that for all $s \in \underline{P}$ we get an irreducible linear group K_s in $\underline{\underline{X}}$ (and by Theorem 4.2.1 and Corollary 4.1.8 there is up to isomorphism only one such) which is abstractly isomorphic to a central product of s copies of Q. Put $\{K_s : s \in \underline{P}\}$ equal to $\underline{\underline{X}}_0$. Since $\underline{\underline{X}}_0$ contains infinitely many isomorphism classes, the closure of $\underline{\underline{X}}_0$ is $\underline{\underline{X}}$. It is convenient to isolate the next step in the proof of Theorem 4.3.1 as a lemma.

4.3.2 LEMMA For each s in $\underline{P}$, K_s has an irreducible linear factor M_s which, when centrally decomposed, is a central product of s critical groups, each of which has derived group of order q.

Proof of Lemma 4.3.2. Suppose $|Q^{(1)}| = q^{\ell}$, $\ell > 1$, and let the central components in a central decomposition of K_s be $Q_1,\ldots,Q_s$. Let g_i and h_i be the generators in a canonical presentation of Q_i, and put $H_i = \langle g_i, h_i^{q^{\ell-1}} \rangle$. Observe that

$$H_i^{(1)} = M(Q_i) = M(K_s), \quad 1 \leq i \leq s,$$

and

$$Z(H_i) = \langle g_i{}^q, [g_i,h_i]^{q^{\ell-1}} \rangle = D(H_i).$$

Put $H_1H_2 \ldots H_s = F_s$. Since $[H_i,H_j] = E$ when $i \neq j$,

$$F_s{}^{(1)} = H_i{}^{(1)} = M(K_s), \ 1 \leq i \leq s.$$

Also,

$$Z(F_s) \geq Z(H_1) \ldots Z(H_s).$$

Suppose that $f = f_1f_2 \ldots f_s \in Z(F_s)$, with $f_i \in H_i$; then for all $x_i \in H_i$,

$$e = [x_i,f] = [x_i,f_i].$$

It follows that $f_i \in Z(H_i)$, $1 \leq i \leq s$, and so

$$Z(F_s) = Z(H_1) \ldots Z(H_s) = F_s{}^q F_s{}^{(1)} = D(F_s).$$

Suppose that V is the $\underline{F}(p)$-space on which K_s acts (faithfully and irreducibly), and let U be an irreducible component of V_{F_s}, which is completely reducible by Maschke's Theorem. Thus

$$V_{F_s} = U \oplus U_1,$$

for some U_1. It follows that

$$V_{M(K_s)} = V_{F_s^{(1)}} = U_{F_s^{(1)}} \oplus U_{1_{F_s^{(1)}}}.$$

But $V_{M(K_s)}$ is homogeneous (Lemma 2.3.3), and so every irreducible component of it is faithful. In particular, the kernel N_s of U avoids $F_s^{(1)}$, and hence by Lemma 2.2.1 (i), $N_s \leq Z(F_s) = D(F_s)$. We now put F_s/N_s equal to M_s; then M_s acts faithfully and irreducibly on U, and so it has cyclic centre. Also, $|M_s^{(1)}| = q$, and so the nonabelian central components in a central decomposition of M_s have derived group of order q. Since $N_s \leq D(F_s)$, $D(M_s) = D(F_s)/N_s$ [9, iii, 3.14], so $|M_s : D(M_s)| = q^{2s}$, and a central decomposition of M_s involves precisely s nonabelian critical subgroups. //

We now continue the proof of Theorem 4.3.1. Since there are only finitely many critical groups in $\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_n$ with derived group of order q, we can repeat our earlier argument, using Lemma 4.3.2. Thus we may as well suppose that $|Q^{(1)}| = q$. Hence by Theorem 4.1.4, either $q = 2$ and $Q \cong R(2)$, or $Q \cong Q(q^k,q)$. If $Q \cong R(2)$ or $Q = Q(q,q)$ with q odd, it follows from Theorem 2.4.5 that $\underline{\underline{V}}$ is $\underline{\underline{A}}_p\underline{\underline{T}}_q$. If $Q \cong Q(2,2)$ recall (Remarks 4.1.5 and 4.1.9) that the central square of Q is isomorphic to a central square of $R(2)$. Hence in this case, we again have by Theorem 2.4.5

that $\underline{\underline{V}}$ is $\underline{\underline{A}}_p \underline{\underline{T}}_2$. Finally, suppose $Q \cong Q(q^k,q)$ for some $k > 1$. We use some of the notation introduced in the proof of Lemma 4.3.2: thus K_s is a central product of $Q_1,\ldots,Q_s$, where $Q_i \cong Q$, and g_i, h_i are the generators in a canonical presentation of Q_i. We recall that

$$Q_i = \langle g_i, h_i : g_i^{q^{k-1}} = [g_i,h_i],\ h_i^{\,q} = e\rangle.$$

By Corollary 4.1.8, we may assume without loss of generality that the amalgamations amount to $g_i^{\,q} = g_1^{\,q}$, $[g_i,h_i] = [g_1,h_1]$, $1 \leq i \leq s$. If $s > 1$, define $f_i = g_i g_1^{-1}$, $2 \leq i \leq s$; then it is easy to see that $\langle f_i, h_i\rangle \cong Q(q,q)$. Put

$$J_s = \langle f_i, h_i : 2 \leq i \leq s\rangle,$$

so that J_s is a normal subgroup of K_s, and is isomorphic to a central product with cyclic centre of $s-1$ copies of $Q(q,q)$. Let V_s be the $\underline{F}(p)$-space on which K_s acts, and let W_s be an irreducible component of $V_{s_{J_s}}$. Observe that the kernel O_s of W_s is a normal subgroup of J_s which avoids $J_s^{(1)}$ (because $J_s^{(1)} = M(J_s) = M(K_s)$). Thus O_s is trivial; that is W_s is faithful. Hence as $\underline{\underline{X}}$ is closed, it contains the restriction of J_s to W_s, for all $s \in \underline{P}$. Theorem 2.4.5 completes the proof. //

We conclude this Chapter with a simple Corollary to Theorem 4.3.1 which characterises $\underline{\underline{A}}_p\underline{\underline{T}}_q$.

4.3.3 COROLLARY The unique jnC subvariety of $\underline{\underline{A}}_{p^k}(\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_{q^\ell})$ is $\underline{\underline{A}}_p\underline{\underline{T}}_q$. //

CHAPTER 5

SOLUBLE JNC VARIETIES

5.1 Introduction

In this section we show how to deduce Theorems A, A*, B and C from Theorem 3.2.1, Corollary 4.3.3, and the following two results.

5.1.1 THEOREM If $\underline{\underline{V}}$ is a soluble jnC variety which is not metabelian, there are primes p, q and r, with p not equal to q or r, and a Cross subvariety $\underline{\underline{N}}$ of $\underline{\underline{V}}$ of exponent a power of r, such that $\underline{\underline{V}} \leq \underline{\underline{A}}_p\underline{\underline{A}}_q\underline{\underline{N}}$.

5.1.2 THEOREM Let p, q and r be primes with p not equal to q or r, and suppose that $\underline{\underline{V}}$ is a jnC subvariety of $\underline{\underline{A}}_p\underline{\underline{A}}_q\underline{\underline{N}}$, where $\underline{\underline{N}}$ is a subvariety of $\underline{\underline{V}}$ of exponent a power r. Then $\underline{\underline{V}}$ is not abelian-by-nilpotent if and only if $\underline{\underline{V}} \leq \underline{\underline{A}}_p\underline{\underline{A}}_q\underline{\underline{A}}_r$.

We observe first that Theorem B is a simple corollary to Theorem 5.1.1 and Corollary 4.3.3. Theorem A follows immediately from Theorems 5.1.1, 5.1.2, and the "internal" result (Theorem A) of John Cossey's thesis [3]. Theorem A* is a consequence of Theorems A and 3.2.1.

In order to establish Theorem C, we must first prove a corollary to Theorem 5 of [15] which Kovács and Newman did not bother to write down.

5.1.3 LEMMA The metabelian jnC varieties which are not abelian are precisely the $\underline{\underline{A}}_p\underline{\underline{A}}_p$, where p is any prime.

Proof. Certainly each $\underline{\underline{A}}_p\underline{\underline{A}}_p$ is metabelian but not abelian. Let $\underline{\underline{V}}$ be a metabelian jnC variety of finite exponent n (say). The Lemma follows from Theorem 5 of [15] if we can show that $\underline{\underline{V}}$ is locally nilpotent. By Theorem 2.4.2 and Corollary 2.4.3 (i), to do this, it suffices to show that the chief-sections of the groups in $\underline{\underline{V}}$ have order at most n^n. Let $G \in \underline{\underline{V}}$, and suppose that H/K is a chief-section of G, say H/K has exponent p. Put $C = \{g : g \in G, [H,g] \leq K\}$; then it is easy to see that C is a normal subgroup of G, and that G/C acts faithfully and irreducibly (by conjugation) on H/K (which we think of as a vector space over $\underline{F}(p)$). But $G^{(1)} \leq C$, since G is metabelian, and so G/C is abelian and hence cyclic. The Lemma follows from [18, 52.24]. //

We can now prove Theorem C. Suppose that $\underline{\underline{V}}$ is a reducible jnC variety which is not decomposable. It follows from Theorem 3.2.1 and Lemma 5.1.3 that $\underline{\underline{V}}$ is soluble but not metabelian. Hence by Theorems 5.1.1 and 5.1.2, there are distinct primes p and q, and a subvariety $\underline{\underline{N}}$ of $\underline{\underline{V}}$ of exponent a power of q, such that $\underline{\underline{V}} \leq \underline{\underline{A}}_p\underline{\underline{N}}$. But then $\underline{\underline{N}}$ must have class at least three (Corollary 4.3.3).

5.2 The Proof of Theorem 5.1.1

As $\underline{\underline{V}}$ is not metabelian, it has finite exponent, say n, and hence it is locally finite. Also, $\underline{\underline{V}}$ is not locally nilpotent (Lemma 3.1.6 (ii)), and so by Lemma 3.1.7, there is a prime, say p, and an infinite set, say Γ, of monolithic groups in $\underline{\underline{V}}$ such that:

(i) $\underline{\underline{V}} = \mathrm{var}\Gamma$;

(ii) the monolith of each group in Γ is complemented, self-centralising, and has exponent p;

(iii) $\{|M(G)| : G \in \Gamma\}$ is an infinite set.

We denote by ℓ the least element of $\underline{P}$ such that $\underline{\underline{V}}$ is a subvariety of $\underline{\underline{A}}^{\ell+1}$; then $\underline{\underline{V}} \wedge \underline{\underline{A}}^{\ell}$ is Cross. (Such an ℓ exists since $\underline{\underline{V}}$ is assumed soluble.) If $\Delta = \{G \in \Gamma : G \in \underline{\underline{A}}^{\ell}\}$, then $\mathrm{var}\Delta$ is a subvariety of $\underline{\underline{V}} \wedge \underline{\underline{A}}^{\ell}$, and hence is Cross. It follows that $\mathrm{var}(\Gamma-\Delta)$ is a non-Cross, and hence equals $\underline{\underline{V}}$, and so we may as well suppose that Δ is empty; that is, we suppose that every group in Γ has solubility length $\ell + 1$ precisely.

Let $G \varepsilon \Gamma$, and denote $M(G)$ by V.

5.2.1 LEMMA $G^{(\ell)} = V$, and if $k < \ell$, $G^{(k)} > V$.

Proof. Since G has solubility length precisely $\ell + 1$, $G^{(k)}$ is trivial if and only if $k \geq \ell + 1$. Since G is also monolithic, $G^{(k)} \geq V$ for $k \leq \ell$. As $G^{(\ell)}$ is abelian, and V is a self-centralising subgroup of $G^{(\ell)}$, $V = G^{(\ell)}$. If $k < \ell$, $G^{(k)} > G^{(\ell)} = V$. //

Let H be a complement for V in G, and observe that by Lemma 5.2.1, $H \varepsilon \underline{\underline{A}}^{\ell}$. Thus if $\underline{\underline{W}} = \mathrm{var}\{H : G \varepsilon \Gamma\}$, $\underline{\underline{W}}$ is a Cross subvariety of $\underline{\underline{V}}$, and $\underline{\underline{V}}$ is a subvariety of $\underline{\underline{A}}_{\underline{\underline{p}}}\underline{\underline{W}}$. By properties (ii) and (iii) of Γ, and [18, 52.24], $\{|H| : G \varepsilon \Gamma\}$ is an infinite set, and hence there is an element m of $\underline{P}$, $0 < m \leq \ell$, which is least such that $\{|H^{(m)}| : G \varepsilon \Gamma\}$ is finite, say $|H^{(m)}| < d$ for all $G \varepsilon \Gamma$. Now H is faithfully and irreducibly represented (by conjugation) on V, which from now on we think of as a vector space over $\underline{F}(p)$.

5.2.2 LEMMA $m \geq \ell - 1$.

Proof. Denote $C_H(H^{(m)})$ by C; then $\{|H : C| : G \in \Gamma\}$ is bounded (by $d!$). If T is a (right) transversal of C in H, and W is an irreducible submodule of V_C, it follows from Clifford's Theorem that

$$V_C = \sum\{Wt : t \in T\},$$

and hence that $\{|W| : G \in \Gamma\}$ is infinite. Since $\underline{\underline{V}}$ is jnC, we have by Theorem 2.4.2 that $\underline{\underline{V}} = \mathrm{var}\{VC : G \in \Gamma\}$. But $C^{(m+1)}$ is trivial, and so $\underline{\underline{V}} \leq \underline{\underline{A}}^{m+2}$; that is, $m \geq \ell - 1$. //

5.2.3 LEMMA There is a prime q, unequal to p, such that either $\underline{\underline{V}}$ is $\underline{\underline{A}}_p\underline{\underline{T}}_q$, or H has an abelian normal subgroup Q of exponent q with $\{|Q| : G \in \Gamma\}$ infinite.

Proof. By Clifford's Theorem, $V_{H^{(\ell-1)}}$ is completely reducible. Since V is faithful and $H^{(\ell-1)}$ is abelian, it follows ([9, v, 5.17]) that $H^{(\ell-1)}$ is a p'-group. Thus in case $m = \ell$ (so that $\{|H^{(\ell-1)}| : G \in \Gamma\}$ is an infinite set), there is a prime, say q, such that $\{|Q| : G \in \Gamma\}$ is infinite, where Q is the socle of the Sylow q-subgroup of $H^{(\ell-1)}$.

Suppose, on the other hand (using Lemma 5.2.2), that $m = \ell - 1$. We can replace H by $C_H(H^{(\ell-1)})$ (as in the proof of Lemma 5.2.2) and apply Lemma 2.4.1 and Theorem 2.4.2 in the usual way. Thus we can suppose that $H^{(\ell-1)}$ is central in H, and in particular that it is cyclic (Lemma 2.3.3). It follows that $H^{(\ell-2)}$ is nilpotent of class two, and so, by the argument used for $H^{(\ell-1)}$ above, it is a p'-group.

In case $\ell = 2$, we use Lemma 2.3.3 and Theorem 2.4.2 to reduce to the case that H is a q-group, for some prime q. It then follows from Corollary 4.3.3 that $\underline{\underline{V}}$ is $\underline{\underline{A}}_p\underline{\underline{T}}_q$.

Finally, if $\ell > 2$, then since $\underline{\underline{V}} \wedge \underline{\underline{A}}^{\ell}$ is Cross, it follows from Theorem 2.4.2 that the orders of the irreducible components of $V_{H^{(\ell-2)}}$ form a finite set as G ranges through Γ. By Lemma 5.2.2, $\{|H^{(\ell-2)}| : G \in \Gamma\}$ is an infinite set, and so there is a prime, say q, such that $\{|A| : G \in \Gamma\}$ is infinite, where A is the Sylow q-subgroup of $H^{(\ell-2)}$. Since $H^{(\ell-1)}$ is cyclic, it follows that $A^{(1)}$ is cyclic. Now V_A is completely reducible by Maschke's Theorem, and so, since V is faithful and $A^{(1)}$ is cyclic, there is an irreducible direct summand, say U, of V_A such that the kernel K of U avoids $A^{(1)}$. Now $VA \in \underline{\underline{V}} \wedge \underline{\underline{A}}^{\ell}$, which is Cross, and so $\{|U| : G \in \Gamma\}$ is a finite set, say $|U| < f$, for all $G \in \Gamma$. Hence $\{|A : K| : G \in \Gamma\}$ is bounded by $f!$. Since $\{|A| : G \in \Gamma\}$ is infinite, it follows that $\{|K| : G \in \Gamma\}$ is infinite. But $K \cap A^{(1)} = E$, and so by Lemma 2.2.1 (i), $K \leq Z(A)$; hence $\{|Z(A)| : G \in \Gamma\}$ is an infinite set, and we can choose Q to be $M(Z(A))$. //

If $\underline{\underline{V}}$ is $\underline{\underline{A}}_p \underline{\underline{T}}_q$, Theorem 5.1.1 is proved, and so we shall assume that H has an abelian normal subgroup Q of exponent q such that $\{|Q| : G \in \Gamma\}$ is infinite. Suppose that V_Q has homogeneous components $U_1, \ldots, U_{a(G)}$, and denote the kernel of U_i by K_i. Now V is faithful, and so $\bigcap_{i=1}^{a(G)} K_i$ is trivial. Since $Q \in \underline{\underline{A}}_q$, Q/K_i is (cyclic) of order q; hence by Lemma 2.2.1 (ii), $\{a(G) : G \in \Gamma\}$ is infinite. Now $a(G)$ is the index of the inertia subgroup of U_1 in G, so every prime dividing $a(G)$ divides n. But n is finite, and so there is a prime, say r (which for all we know at this stage could be p), such that $\{a_r(G) : G \in \Gamma\}$ is infinite, where $a_r(G)$ is the r-share of $a(G)$. If R is a Sylow r-subgroup of G, and W is an irreducible submodule of V_{QR}, it follows from Lemma 2.3.2 that if W_Q has $d(G)$ homogeneous components, then $d(G) \geq a_r(G)$. In particular, since

$\underline{\underline{V}}$ is jnC, it follows from Theorem 2.4.2 that $\underline{\underline{V}} = \mathrm{var}\{WQR : G \in \Gamma\}$. Recall that since $\underline{\underline{V}}$ is not locally nilpotent, Corollary 2.4.3 (i) implies that there is a bound, say c, on the nilpotency class of the nilpotent groups in $\underline{\underline{V}}$. Hence

$$\underline{\underline{V}} \leq \underline{\underline{A}}_p \underline{\underline{A}}_q \ \mathrm{var}\{R : G \in \Gamma\} ,$$

and $\mathrm{var}\{R : G \in \Gamma\}$ is a (Cross) subvariety of $\underline{\underline{V}}$ of class c and exponent a power of r.

Since $\{d(G) : G \in \Gamma\}$ is infinite, there is a group, say G_1, in Γ such that $d(G_1) > p^c$. Lemma 2.3.4 now shows that r is not equal to p, and the proof of Theorem 5.1.1 is complete. //

5.3 The proof of Theorem 5.1.2

The "if" part of Theorem 5.1.2 is trivial, so suppose that $\underline{\underline{V}}$ is not abelian-by-nilpotent. Then $\underline{\underline{V}} \wedge \underline{\underline{A}}_p\underline{\underline{N}}$ and $\underline{\underline{V}} \wedge \underline{\underline{A}}_q\underline{\underline{N}}$ are Cross subvarieties of $\underline{\underline{V}}$, and q is not equal to r.

Observe that $\underline{\underline{V}}$ is locally-finite, being soluble of finite exponent. Now $\underline{\underline{N}}$ is a Cross variety, and so by Theorem 2.4.2, there is an integer, say c, such that the nilpotent groups in $\underline{\underline{N}}$, and hence also those in $\underline{\underline{V}}$, have class at most c. It follows from Corollary 2.4.3 (ii) that $\underline{\underline{V}}$ is not locally nilpotent. By Lemma 3.1.7, there is a prime, say t, and an infinite set, say Γ, of monolithic groups in $\underline{\underline{V}}$ such that

(i) $\underline{\underline{V}} = \text{var}\Gamma$;

(ii) the monolith of each group in Γ is complemented, self-centralising, and has exponent t;

(iii) $\{|M(G)| : G \in \Gamma\}$ is an infinite set.

In every group in $\underline{\underline{V}}$, the Sylow p-subgroup is normal, and its quotient group lies in $\underline{\underline{A}}_q\underline{\underline{N}}$. Since $\underline{\underline{V}} \wedge \underline{\underline{A}}_q\underline{\underline{N}}$ is Cross, it follows from Theorem 2.4.2 that

$$t = p.$$

Suppose $G \in \Gamma$, and denote the monolith $M(G)$ of G by V. Let H be a Hall p'-subgroup of G, and let Q be a Sylow q-subgroup and R a Sylow r-subgroup of H. Denote $M(Z(R))$ by Y. Then V can be thought of as a vector space over $\underline{F}(p)$, in which case H is faithfully and irreducibly represented (by conjugation) on V. Suppose that $U_1, \ldots, U_{a(G)}$ are the homogeneous components of V_Q, and let K_i be the kernel of U_i. We need to prove three preparatory lemmas.

5.3.1 LEMMA The set $T = \{|Q| : G \in \Gamma\}$ is infinite.

Proof. By property (iii) of Γ, $\{|V| : G \varepsilon \Gamma\}$ is infinite. As V is a self-centralising chief-section of G (property (ii) of Γ), it follows from [18, 52.24] that $\{|H| : G \varepsilon \Gamma\}$ is infinite. Suppose contrary to the Lemma that T is finite, say $|Q| < u$ for all G in Γ, and denote $C_H(Q)$ by C. Then $\{|H : C| : G \varepsilon \Gamma\}$ is bounded by $u!$, and so by Clifford's Theorem (part (i)), if W is an irreducible submodule of V_C, $\{|W| : G \varepsilon \Gamma\}$ is infinite. Using Lemma 2.4.1, we replace WC by $(WC)^*$; by Theorem 2.4.2, $\underline{\underline{V}} = \mathrm{var}\{(WC)^* : G \varepsilon \Gamma\}$. Thus we may as well suppose that $C = H$, for all G in Γ. In this case, of course, $Q \leq Z(H)$, and so Q is cyclic of order at most q. It follows that R is normal in H, and $|H : R| \leq q$. Then again by Clifford's Theorem (and Lemma 2.3.3), V_R is a (homogeneous) direct sum of at most q irreducible submodules of common dimension. Thus there is no bound on the orders of the irreducible submodules of V_R, as G ranges through Γ. It follows from Theorem 2.4.2 that $\underline{\underline{V}} = \mathrm{var}\{VR : G \varepsilon \Gamma\}$, and hence that $\underline{\underline{V}}$ is abelian-by-nilpotent, a contradiction. //

5.3.2 COROLLARY The set $\Phi = \{|R| : G \varepsilon \Gamma\}$ is infinite.

Proof. As Q/K_i is faithfully and irreducibly represented on an irreducible component of U_i, it is cyclic of order q. But V is faithful, and so $\bigcap\{K_i : 1 \leq i \leq a(G)\}$ is trivial. As T is infinite (Lemma 5.3.1), it follows from Lemma 2.2.1 (ii) that $\{a(G) : G \in \Gamma\}$ is infinite. By Clifford's Theorem, $a(G) \leq |H : Q| = |R|$, and so Φ is infinite. //

5.3.3 LEMMA The set $\Psi = \{|Y| : G \in \Gamma\}$ is infinite.

Proof. By Maschke's Theorem, V_R is completely reducible; let

$$V_R = V_1 \oplus \cdots \oplus V_{\ell(G)}$$

be a direct decomposition of V_R into irreducible submodules. Since $\underline{\underline{V}} \wedge \underline{\underline{A}}_p \underline{\underline{N}}$ is Cross, it follows from Theorem 2.4.2 that $\{|V_i| : 1 \leq i \leq \ell(G), G \in \Gamma\}$ is finite, say $|V_i| < m$ for all i, G. If the kernel of V_i is R_i, then

$$\{|R : R_i| : 1 \leq i \leq \ell(G), G \in \Gamma\}$$

is bounded by $m!$. Since V is faithful, $\bigcap\{R_i : 1 \leq i \leq \ell(G)\}$ is trivial. Choose a subset, say $\Lambda(G)$, of $\{1,\ldots,\ell(G)\}$ which is minimal with respect to $\bigcap\{R_i : i \in \Lambda(G)\}$ being trivial. Since Φ is an infinite set (Corollary 5.3.2), it follows from

Lemma 2.2.1 (ii) that $\{|\Lambda(G)| : G \in \Gamma\}$ is an infinite set. We suppose that the components of V_R have been numbered so that $\Lambda(G) = \{1,\ldots,k(G)\}$. Put

$$S_i = \cap\{R_j : 1 \leq j \leq k(G),\ j \neq i\},\quad 1 \leq i \leq k(G).$$

By the minimality of $\Lambda(G)$, each S_i is a nontrivial normal subgroup of R, and if $i \neq j$, $S_i \cap S_j = E$. Hence

$$|Y| \geq 1 + \sum_{i=1}^{k(G)} (|S_i \cap Y| - 1).$$

But for each i, $S_i \cap Y$ is nontrivial, so $|Y| > k(G)$. Since $k(G) = |\Lambda(G)|$, and $\{|\Lambda(G)| : G \in \Gamma\}$ is infinite, the proof is complete. //

We are now in a position to prove Theorem 5.1.2. Observe that QY is a normal subgroup of H which is supplemented in H by R. Thus if W is an irreducible submodule of V_{QY}, it follows from Clifford's Theorem that

$$V_{QY} = \sum\{ Wx : x \in R\}.$$

Suppose that M is the kernel of W; then the kernel of Wx is M^x. Since V is faithful, it follows that

$$\bigcap\{M^x : x \in R\} = E.$$

But if $x \in R$,

$$M^x \cap Y = (M \cap Y)^x = M \cap Y,$$

and so

$$M \cap Y = E.$$

Thus Y acts faithfully on W; that is, Y is isomorphic to a subgroup of $\mathrm{Aut}W$. In particular, $|W|! > |Y|$. It follows from Lemma 5.3.3 that $\{|W| : G \in \Gamma\}$ is an infinite set, and so by Theorem 2.4.2,

$$\underline{\underline{V}} = \mathrm{var}\{WQY : G \in \Gamma\}.$$

Hence $\underline{\underline{V}}$ is a subvariety of $\underline{\underline{A}}_p\underline{\underline{A}}_q\underline{\underline{A}}_r$. //

CHAPTER 6

GAPS

6.1 Irreducible jnC varieties

We begin this section by determining the irreducible Cross varieties.

6.1.1 LEMMA A variety is generated by a finite simple group if and only if it is irreducible and Cross.

Proof. The "only if" part is trivial. Suppose $\underline{\underline{V}}$ is an irreducible Cross variety, say $\underline{\underline{V}} = \mathrm{var}G$, where G is a finite group. Let $G_0 \triangleright \ldots \triangleright G_k = E$ be a composition series for G, and observe that $\underline{\underline{V}}$ is a subvariety of $\prod_{i=0}^{k-1} \mathrm{var}(G_i/G_{i+1})$. Since $\underline{\underline{V}}$ is irreducible, it follows that for some i, $\underline{\underline{V}} = \mathrm{var}(G_i/G_{i+1})$. //

6.1.2 THEOREM An irreducible locally finite variety either (a) is generated by a finite simple group, or (b) is locally nilpotent but insoluble, or (c) contains infinitely many (isomorphism classes of) finite simple groups.

Proof. (P. Hall and Graham Higman [5], especially the proof of 4.4.1; L.G. Kovács [12], especially page 13). We shall prove that if

$\underline{\underline{V}}$ is a locally finite variety for which (a), (b) and (c) do not hold, then $\underline{\underline{V}}$ is reducible. Observe first that a soluble variety of finite exponent is reducible unless it is an $\underline{\underline{A}}_p$, in which case, of course, it is generated by a finite simple group. By Lemma 6.1.1, a Cross variety is reducible unless it is generated by a finite simple group. Thus we are left to show that $\underline{\underline{V}}$ is reducible if it is locally finite, not locally nilpotent, not Cross, and contains only finitely many (isomorphism classes of) finite simple groups.

Suppose $\underline{\underline{V}}$ has exponent (precisely) n, and let $p_1,\ldots,p_r$ be the distinct primes dividing n. Since $\underline{\underline{V}}$ is locally finite but not locally nilpotent, $r > 1$. Put $\underline{\underline{V}} \wedge \underline{\underline{B}}_{n/p_i}$ equal to $\underline{\underline{V}}_i$, $1 \leq i \leq r$, and denote $\underline{\underline{V}}_1 \vee \ldots \vee \underline{\underline{V}}_r$ by $\underline{\underline{W}}$. The $\underline{\underline{V}}_i$ are pairwise-incomparable on account of their exponents, so they are all proper subvarieties of $\underline{\underline{W}}$. Since $\underline{\underline{W}}$ is a subvariety of $\prod_{i=1}^{r} \underline{\underline{V}}_i$, it is reducible; thus we may suppose that $\underline{\underline{W}}$ is a proper subvariety of $\underline{\underline{V}}$.

Let Λ be a (finite) set containing one copy of each of the finite simple groups in $\underline{\underline{V}}$, and put $\underline{\underline{X}} = \mathrm{var}\{\mathrm{Aut}B : B \in \Lambda\}$. Now $\underline{\underline{X}}$ is Cross, and hence so is $\underline{\underline{V}} \wedge \underline{\underline{X}}$; it follows that $\underline{\underline{V}} \wedge \underline{\underline{X}}$ is a proper subvariety of $\underline{\underline{V}}$.

To complete the preparations, let $\underline{\underline{R}}$ be the class consisting of the finite nilpotent groups in $\underline{\underline{V}}$. Since $\underline{\underline{V}}$ is locally finite but not locally nilpotent, the locally nilpotent groups in $\underline{\underline{V}}$ form a proper subvariety $\underline{\underline{V}}_{LN}$ of $\underline{\underline{V}}$ (c.f. (2) of [11]), and $\underline{\underline{R}} \subseteq \underline{\underline{V}}_{LN}$. If $\underline{\underline{S}}$ is the class consisting of all the finite soluble groups in $\underline{\underline{V}}$, then by [12, page 13(3)], there is a positive integer, say k, such that $\underline{\underline{S}}$ is a subclass of $\underline{\underline{R}}^k$. In particular, $\underline{\underline{S}}$ is a subclass of $\underline{\underline{V}}_{LN}{}^k$.

We shall show that $\underline{\underline{V}}$ is a subvariety of $\underline{\underline{V}}_{LN}{}^k.(\underline{\underline{V}} \wedge \underline{\underline{X}}).\underline{\underline{W}}$.

Since $\underline{\underline{V}}$ is locally finite, it is generated by its finite groups [18, 15.61]. The soluble radical of a finite group in $\underline{\underline{V}}$ belongs to $\underline{\underline{V}}_{LN}{}^k$, and its factor group, if nontrivial, is a subdirect product of monolithic groups in $\underline{\underline{V}}$ with nonabelian monoliths. Thus the Theorem is proved if we can show that a finite monolithic group G in $\underline{\underline{V}}$ with nonabelian monolith belongs to $\underline{\underline{XW}}$.

Denote $M(G)$ by M, and suppose that the (necessarily nonabelian) simple direct factors of M are $M_1,\ldots,M_s$: observe that they form a complete conjugacy class in G. Denote $N_G(M_i)$ by N_i, $C_G(M_i)$ by C_i, and $\cap\{N_i : 1 \leq i \leq s\}$ by N. If $C = \cap\{C_i : 1 \leq i \leq s\}$, then C is a normal subgroup of G which avoids M (for the centre of M is trivial); hence $C = E$. Now $N \cap C_i$ is normal in N, and so by Lemma 2.2.1 (ii),

N is isomorphic to a subgroup of $N/N \cap C_1 \times \ldots \times N/N \cap C_s$. But $N/N \cap C_i$ is isomorphic to a subgroup of $\mathrm{Aut}M_i$, and so $N \in \underline{\underline{X}}$.

We are thus left to prove that $G/N \in \underline{\underline{W}}$. In fact we shall show that if p divides $|M_1|$, then $G/N \in \underline{\underline{V}} \wedge \underline{\underline{B}}_{n/p} \leq \underline{\underline{W}}$.

By Corollary 2.2.6, G is represented (by conjugation) as a transitive permutation group on $\{M_i : 1 \leq i \leq s\}$ with kernel N; the stabiliser of the "point" M_i being N_i/N. Let p be any prime which divides $|M_1|$, and suppose that p^f is the p-share of the exponent of G/N. Choose any p-element g of G such that $|gN| = p^f$, and let P be a Sylow p-subgroup of G which contains g. If $|g|$ is greater than p^f, $G/N \in \underline{\underline{B}}_{n/p} \wedge \underline{\underline{V}}$, and the Theorem is proved. We shall suppose therefore that $|g| = p^f$, and find an element $b \in P$ such that $|gb| > p^f$. Since gN has at least one orbit of cardinality p^f, we may assume without loss of generality that the orbit containing M_1 is one such and that

$M_1^{g^i} = M_{1+i}$, $0 \leq i < p^f$. As $P \cap M$ is a Sylow p-subgroup of M, and $P \cap M_1$ is a Sylow p-subgroup of M_1, we have $P \cap M_1 > E$. If b is any nonidentity element of $P \cap M_1$, gb is a p-element of G, and

$$(gb)^{p^f} = g^{p^f} b^{g^{p^{f-1}}} \ldots b^{g^2} b^g b \neq e,$$

since $g^{p^f} = e$, and the other factors of the product are nontrivial and belong to pairwise-distinct factors of the given direct decomposition of M. //

6.1.3 COROLLARY Let $\underline{\underline{V}}$ be a jnC variety. Then $\underline{\underline{V}}$ is irreducible if and only if either (a) $\underline{\underline{V}}$ is not locally finite, or (b) $\underline{\underline{V}}$ is locally finite and locally nilpotent but insoluble, or (c) $\underline{\underline{V}}$ is locally finite and contains infinitely many (isomorphism classes of) finite simple groups.

Proof. The "only if" part follows immediately from Theorem 6.1.2. The "if" part is a consequence of Lemma 3.1.6. //

As far as irreducible jnC varieties of type (c) are concerned, we shall be content (at least in this thesis) to say nothing more than Higman has said already (see section 1.2).

We recall that the result of Bachmuth, Mochizuki and Walkup [1] states that $\underline{\underline{K}}_5$ is insoluble, and hence that it is non-Cross. Since the jnC subvarieties of $\underline{\underline{K}}_5$ are all insoluble [15, Theorem 5], we conclude that irreducible jnC varieties of type (b) exist. In fact there may well be infinitely many such irreducibles, as it seems reasonable to expect that $\underline{\underline{K}}_p$ will turn out to be insoluble for almost all primes. We do not imply that if $\underline{\underline{K}}_p$ is non-Cross then it necessarily has a unique jnC subvariety, although one is tempted to conjecture that it has, and that if Ω is the set of all such jnC varieties, $\Omega \cup \{\underline{\underline{A}}_p\underline{\underline{A}}_p : p$ any prime$\}$ contains every locally finite, locally nilpotent jnC variety.

If $\underline{\underline{K}}_p$ turned out to be Cross for some prime p greater than 4381, the negative solution by P.S. Novikov and S.I. Adyan [19] to the Burnside Conjecture for such primes would imply the existence of irreducible jnC varieties of type (a). Again, problem 5 of Hanna Neumann's book [18, page 42] asks "does there exist a nonabelian variety, all of whose finite groups are abelian?" If the answer were "Yes, $\underline{\underline{V}}$ is one of finite exponent", then a subvariety of $\underline{\underline{V}}$ minimal with respect to being nonabelian would be jnC and not locally finite.

Let us consider the implications of the existence of a non locally finite jnC variety $\underline{\underline{V}}$ of finite exponent n (say). If $\underline{\underline{V}}$ is generated by its finite groups, then the restricted Burnside Conjecture for exponent n must be false, and so the class $\underline{\underline{K}}_n$ cannot constitute a variety [11]. Alternatively, if $\underline{\underline{V}}$ is not generated by its finite groups, then it has only finitely many subvarieties, whereas this is a property which one might hope only Cross varieties have. Evidence such as this seems to indicate that it is a very difficult problem to decide whether or not there exist irreducible jnC varieties of type (a) of finite exponent.

We conclude this section with a lemma which may be useful in attempts to classify irreducible jnC varieties of type (b).

6.1.4 LEMMA Let $\underline{\underline{V}}$ be a locally finite, locally soluble jnC variety. Then $\underline{\underline{V}}$ is insoluble if and only if it contains a set $\{G_\ell : \ell \in \underline{P}\}$ of finite groups, such that for all $\ell \in \underline{P}$:

(i) G_ℓ is soluble of length precisely ℓ;

(ii) every proper section of G_ℓ belongs to $\underline{\underline{A}}^{\ell-1}$; in particular, G_ℓ is critical;

(iii) G_ℓ is isomorphic to a proper section of $G_{\ell+1}$.

Proof. The "if" part is trivial, so suppose that $\underline{\underline{V}}$ is insoluble. Then $\underline{\underline{V}} \wedge \underline{\underline{A}}^\ell$ is a Cross subvariety of $\underline{\underline{V}}$ for all ℓ in $\underline{P}$. Since $\underline{\underline{V}}$ is locally finite and locally soluble, it is generated by its finite soluble groups [18, 15.61], and so $\underline{\underline{V}} \wedge \underline{\underline{A}}^\ell$ is a proper subvariety of $\underline{\underline{V}} \wedge \underline{\underline{A}}^{\ell+1}$, for all ℓ in $\underline{P}$. Let Γ_ℓ be a (finite) set containing one copy of each (isomorphism type of) critical group of solubility length precisely ℓ in $\underline{\underline{V}} \wedge \underline{\underline{A}}^\ell$, and denote $\cup\{\Gamma_\ell : \ell \in \underline{P}\}$ by Γ. If $G, H \in \Gamma$, write $G \vdash H$ in case G is isomorphic to a proper section of H. A totally ordered (by $\vdash$) subset of Γ is called a chain, and a subset Δ of Γ is <u>closed</u> if given $H \varepsilon \Delta$ and $G \vdash\!\!\!-\!\!- H$, it follows that $G \varepsilon \Delta$. Let Λ be a closed chain in Γ such that $\Lambda \cap \Gamma_\ell$ is a proper subset of Λ for all ℓ in $\underline{P}$. Since Λ is closed, for all ℓ in $\underline{P}$, we can find a group, say G_ℓ, in $\Lambda \cap \Gamma_\ell$ such that every proper section of G_ℓ belongs to $\underline{\underline{A}}^{\ell-1}$. //

I have recently been trying to prove that if $\underline{\underline{V}}$ is a locally soluble jnC variety of exponent dividing p^2, then either $\underline{\underline{V}}$ is soluble (and hence is $\underline{\underline{A}}_p\underline{\underline{A}}_p$), or $\underline{\underline{V}}$ has exponent p. I tried to go about this in the following way: suppose $\underline{\underline{V}}$ is insoluble, and let $\{G_\ell : \ell \varepsilon \underline{P}\}$ be as in the statement of Lemma 6.1.4. Then $\underline{\underline{V}}$ has exponent p if there is an unbounded function $\alpha : \underline{P} \to \underline{P}$ such that $\mathrm{var}G_\ell \wedge \underline{\underline{K}}_p$ contains a soluble group of solubility length precisely $\alpha(\ell)$. At this stage α could even be the identity function, as I don't know whether or not there can be a G_3 such that $\mathrm{var}G_3 \wedge \underline{\underline{K}}_p$ is metabelian.

6.2 Reducible jnC varieties

We recall that the reducible jnC varieties which have still to be classified are the indecomposables (if any exist). According to Corollary 1.2.3, there are none if and only if Conjecture 1.2.2 is true. As we noted in section 1.2, however, the only significant progress we have made so far with Conjecture 1.2.2 is Theorem B, which shows it to be true in case $\underline{\underline{N}}$ has class two. I first thought that the best way to tackle Conjecture 1.2.2 was to show it true in case $\underline{\underline{N}}$ is a subvariety of $\underline{\underline{N}}_3 \wedge \underline{\underline{B}}_q$, and then to prove that to do this is in fact sufficient. Why? Because I tend to believe that the Conjecture is true, and I feel that the difficulties posed by the general problem are essentially present in that special case. Let me say straightaway that I haven't the slightest idea how to achieve the cut-down to $\underline{\underline{N}} \leq \underline{\underline{N}}_3 \wedge \underline{\underline{B}}_q$. Consider the following conjecture:

CONJECTURE (q) There is an unbounded function $\alpha : \underline{P} \to \underline{P}$, such that if $G \in \underline{\underline{N}}_3 \wedge \underline{\underline{B}}_q$, $\underline{\underline{N}}_2(G)$ is cyclic, and $d(G) = n$, then the index of $G^{(1)}$ in any subgroup of G maximal with respect to being nilpotent of class two is at least $q^{\alpha(n)}$.

The relevance of Conjecture (q) is that if it is true, then I can prove Conjecture 1.2.2 under the additional hypothesis $\underline{\underline{N}} \leq \underline{\underline{N}}_3 \wedge \underline{\underline{B}}_q$. As the proof of this statement is long, and is very similar to the proofs given in Chapter 5, it is omitted.

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